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Planar scattering amplitudes from Wilson loops

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Abstract: We derive an expression for parton scattering amplitudes of planar gauge theory in terms of sums of Wilson loops. We study in detail the example of Yang-Mills theory with an adjoint Higgs field. The expression exhibits the T-duality performed by Alday and Maldacena in the AdS dual as a Fourier transform in loop space. When combined with the AdS/CFT correspondence for Wilson loops and a strong coupling argument for the dominance of 1PI diagrams, this leads to a derivation of the Alday-Maldacena holographic prescription for scattering amplitudes in terms of momentum Wilson loops. The formula leads to a conjecture for a relationship between position-space and momentum-space Wilson loops in $\mathcal{N} = 4$ SYM at finite coupling.

Keywords: 1/N Expansion, AdS-CFT Correspondence, QCD.

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Contents

1. Introduction

Imagine that you are so powerful that given an arbitrary gauge theory, you can calculate the expectation value of the Wilson loop around an arbitrary path, as a functional of the path. This would be a lot of information about the gauge theory. In particular, it should determine the scattering amplitudes of the partons of the theory. How would one extract this information?

In this paper we construct a prescription which extracts the parton scattering amplitudes from Wilson loop expectation values. The result simplifies dramatically in the 't Hooft limit, and we focus on this case. A formula similar to our result was conjectured by Polyakov, based on string theory intuition [1]. That formula (equation (2.5) below), literally applies to interactions mediated by a scalar field. Much of our effort in this paper will be devoted to generalizing this for interactions mediated by a gauge field.

The immediate motivation for our work was the connection between parton scattering amplitudes and Wilson loops exposed by the work of Alday and Maldacena (AM) [2–4]. Our results give a partial derivation of the Alday-Maldacena prescription in terms of previously-understood entries in the AdS/CFT dictionary.

Our work may shed light on the mysterious relation between position space loops and momentum space loops suggested by the work of [2]. Specifically, matching our prescription to the correctness of the AM result for gluon scattering at large λ, combined with the weak coupling relation between gluon scattering and position space Wilson loops, suggests a conjecture for the precise relationship, elaborated in section 5.

It may also teach us something about how a string theory emerges from the gauge theory. One of Polyakov's stated motivations for studying the formula (2.5) was to understand the action of the loop operator in the string language [5]. We comment on this at the end.

The use of Wilson loops in the study of scattering has a long history. Attempts to reformulate gauge theory in terms of loop space [6–12] have occasionally resulted in related formulae, none of which precisely met our needs. We were unable to find in the literature an answer to the question posed in the first paragraph, namely an expression for the scattering amplitude as a sum over Wilson loops. Previous attempts to relate scattering amplitudes to Wilson loops in string theory using the eikonal approximation include [13].

The paper is organized as follows. In the next section we begin our study of planar gauge theory scattering amplitudes and make a first pass at a worldline description. Then we show that the mysterious-seeming T-duality used by Alday and Maldacena in finding their saddle point has a simple interpretation in terms of a Fourier transform in loop space [14]. In section four, we give a systematic treatment. In section five, we discuss the most important difference between the heuristic (2.5) and the correct formula, namely, the fact that the Wilson sums give only the 1PI effective action, which must be connected in trees to get the full scattering amplitude. In section six we work out the prescription in detail in an example. After that we discuss some possible further applications of the prescription. Sequestered to the three appendices are our discussions of reparametrization-invariant worldline theories (A), worldline superspace (B), and the action of the loop operator (C).

2. Gluon scattering amplitudes

A well known question in QED is the following: The electron field operator is not gauge invariant. However, there are physical states in the theory where an electron is localized around some point $\mathbf{x}$. So, what is an operator that creates from the vacuum a state with an electron at the point $\mathbf{x}$? An example of such an operator is the electron field operator attached to a Wilson line from the point $\mathbf{x}$ to infinity. This operator is charged only under the global part of the gauge group, which is not gauged; this global charge is the electron charge. Given such an operator, one can then consider its Green functions and obtain scattering amplitudes from them via an LSZ formula. A consistency requirement of

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such manifestly gauge invariant definition of electron scattering amplitudes is that it will reproduce the known perturbative expansion of the amplitude.

In this paper, we will derive a gauge invariant expression for *planar* parton scattering amplitudes in non-abelian gauge theory. Similarly to electrons in QED, these are charged only under the global part of the gauge group. Although the basic block in our planar expression will be the Wilson loop, it also can be expanded as a reorganization of planar Feynman graphs and therefore it will automatically reproduce the perturbative expansion. In this subsection we start with an intuitive guess for such an expression. This cartoon of the formula does not reproduce the known perturbative expansion of the amplitude. However, it captures a lot of the physics of the problem and will be presented first to develop intuition.

A gluon is characterized by its momentum $\mathbf{k}_i$, its polarization vector ε_i and an adjoint color matrix T^{a_i}. In perturbation theory, a gluon scattering amplitude can be rearranged as a sum over all possible color contractions of the external gluons. The coefficients of the color traces are independent of the specific color matrices in the trace and are called *partial amplitudes*. In the planar limit, only single traces contribute and the amplitude takes the form

$$\mathcal{A}_n^{\text{planar}} = \sum_\pi \operatorname{tr}\left(T^{\pi(1)} \dots T^{\pi(n)}\right) A_n((\mathbf{k}_{\pi(1)}, \varepsilon_i), \dots, (\mathbf{k}_{\pi(n)}, \varepsilon_n)) \ . \tag{2.1}$$

In this paper we will be interested only in these color-ordered partial amplitudes.

2.1 A first pass at scattering amplitudes from Wilson loops

Based on string theory intuition, Polyakov [1] suggested the following expression for the gluon partial amplitude in QCD

$$A_n^P \equiv \int [D\mathbf{x}] \prod_i \int_{s_{i-1}} ds_i \, \varepsilon_i^\mu \frac{dx_\mu}{ds} e^{i\mathbf{k}_i\cdot\mathbf{x}(s_i)} \langle W[\mathbf{x}(\cdot)]\rangle \,, \tag{2.2}$$

where

$$\int [D\mathbf{x}] \cdots \equiv \int_0^\infty \frac{dT}{T} \mathcal{N} \int [D\mathbf{x}]_1 \, e^{-\frac{1}{2}\int_0^T \dot{\mathbf{x}}^2 ds} \dots \tag{2.3}$$

is the integral over all closed loops with measured with respect to the induced metric on the worldline [14]. The integral $\int [D\mathbf{x}]_1$ in (2.3) is an integral over all closed curves defined with respect to a trivial worldline metric. That is, it is defined by dividing the segment $[0, T]$ into infinitesimal pieces which are equally-spaced in the parameter s:

$$\int [D\mathbf{x}]_1 \equiv \prod_\ell d^D x(s_\ell) \,, \tag{2.4}$$

where $D = 4$ is the number of spacetime dimensions. The normalization constant $\mathcal{N}$ is defined such that in the continuum limit

$$\mathcal{N} \int [D\mathbf{x}(\cdot)]_1 e^{-\frac{1}{2}\int_0^T \dot{\mathbf{x}}^2 ds} = [2\pi T]^{-D/2} \ .$$

As in open string theory, the terms

$$\varepsilon_i^\mu \frac{dx_\mu}{ds} e^{i\mathbf{k}_i\cdot\mathbf{x}(s_i)}$$

are gluon like vertex insertion ordered and integrated along the loop. Each closed loop is weighted by the expectation value of corresponding Wilson loop $\langle W[x(\cdot)]\rangle$.

From the open string point of view, the loop represent the boundary of an open string and the Wilson loop expectation value represent the worldsheet path of an open string (embedded in one higher dimension) with a fixed boundary loop. So equation (2.2) stands somewhere between the gauge theory and the string theory descriptions where we separate the dynamics of the boundary of the open string from the dynamics of its bulk, represented in the gauge theory languish by the Wilson loop expectation value.

The generalization of (2.2) for finite worldline mass m is

$$A_n^{\mathbf{P}} = \int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}(\cdot)]_1 e^{-\int_0^T ds\left(\frac{1}{2}\dot{\mathbf{x}}^2+m^2\right)} F[\mathbf{x}(\cdot);\{\varepsilon_i,\mathbf{k}_i\}]\,, \tag{2.5}$$

where

$$F[\mathbf{x}(\cdot);\{\varepsilon_i,\mathbf{k}_i\}] = \prod_i \int_{s_{i-1}}^{T} ds_i\, \varepsilon_i \cdot \dot{\mathbf{x}}(s_i) e^{i\mathbf{k}_i\cdot\mathbf{x}(s_i)} \langle W[\mathbf{x}(\cdot)]\rangle \ .$$

$F[\mathbf{x}(\cdot);\{\varepsilon_i,\mathbf{k}_i\}]$ is reparametrization invariant. Therefore, $A_n^{\mathbf{P}}$ can also be written in terms of a Nambu-Goto-like action [14]

$$\begin{aligned} A_n^{\mathbf{P}} &= \int \frac{[D\mathbf{x}(\cdot)]_{\dot{\mathbf{x}}^2}}{[Df]} e^{-m_0\int_0^1\sqrt{\dot{\mathbf{x}}^2}} F[\mathbf{x}(\cdot);\{\varepsilon_i,\mathbf{k}_i\}] \\ &= \int \frac{[D\mathbf{x}(\cdot)]_{\dot{\mathbf{x}}^2}}{[Df]} e^{-m_0\int_0^1\sqrt{\dot{\mathbf{x}}^2}} \prod_i \int_{s_{i-1}}^1 ds_i\, \varepsilon_i\cdot\dot{\mathbf{x}}(s_i) e^{i\mathbf{k}_i\cdot\mathbf{x}(s_i)}\langle W[\mathbf{x}(\cdot)]\rangle\,, \end{aligned} \tag{2.6}$$

where $\int [D\mathbf{x}(\cdot)]_{\dot{\mathbf{x}}^2}$ is an integral over all closed curves with respect to the induced metric on the worldline, $[Df]$ stands for the volume of the gauge (reparametrization) group and m_0 is the bare mass which is dialed such that (2.6) admits a continuum limit with a physical mass m.

We can interpret A_n^P as follows: Inserting a spacelike Wilson loop, even a smooth one, into a Lorentzian gauge theory is a huge perturbation of the vacuum. Since the loop is a color source with short-distance structure, there is a large amplitude to create many very hard gluons, and hence a very small amplitude not to do so. In the formula for the scattering amplitude in terms of Wilson loops, the integral over contours constructs a superposition of loops which create a fixed number of external gauge bosons.

The expression for the gluon amplitude in terms of closed Wilson loops (2.2) has a natural generalization for scattering of quarks. In this paper however, we will discuss the scattering of adjoint fields only.

Equation (2.2) is roughly of the form that we will derive at large N using the worldline description of one loop determinants. We label it with a **P** to distinguish it from the correct formula which we will eventually write. As we will see, although the formal expression (2.2) captures much of the physics of the amplitude, it is not correct for several reasons. First, even in pure YM theory, the Wilson loop is the phase acquired by a scalar field, not a vector field. Accounting correctly for the spin modifies both the "gluon vertex operator" and the form of the Wilson loop. Second, the amplitude has IR divergences and requires regularization. As we will discuss, one way to regularize the amplitude is by turning on a worldline mass (2.5). Thirdly, an equation in the spirit of (2.2) holds only in the planar limit and in general gets (tractable) $1/N$ corrections. Finally, we will see that even at large N the correct formula is a sum over tree-level Feynman diagrams using objects like (2.2) as vertices.

To see that some correction to $A^{\mathbf{P}}$ will be needed, even perturbatively, note that to lowest order in the 't Hooft coupling λ, it gives only a scalar one loop contribution to the amplitude [15], while any tree level contribution is absent. Non-perturbatively, we can see the need for improvement as follows. The amplitude A_n is invariant under shift of a polarization vector by the corresponding momenta, (as is necessary for gauge invariance)

$$A_n\left[\ldots,(\mathbf{k}_i,\varepsilon_i+c\,\mathbf{k}_i),\ldots\right]=A_n\left[\ldots,(\mathbf{k}_i,\varepsilon_i),\ldots\right]\,, \tag{2.7}$$

where c is a constant parameter. In $A^{\mathbf{P}}$ (2.2), we note that a longitudinally-polarized gluon vertex

$$\mathbf{k}\cdot\frac{d\mathbf{x}}{ds}\,e^{i\mathbf{k}\cdot\mathbf{x}(s)}=-i\frac{d}{ds}\,e^{i\mathbf{k}\cdot\mathbf{x}(s)}$$

is a total derivative. However, in the partial amplitude, s_i is integrated only on the segment $[s_{i-1},s_{i+1}]$. Therefore, instead of (2.7) we find

$$\begin{aligned}A_n^{\mathbf{P}}&\left[\ldots,(\mathbf{k}_i,\varepsilon_i+c\,\mathbf{k}_i),\ldots\right]-A_n^{\mathbf{P}}\left[\ldots,(\mathbf{k}_i,\varepsilon_i),\ldots\right]\\ &=\;icA_{n-1}^{\mathbf{P}}\left[\ldots,(\mathbf{k}_{i-1}+\mathbf{k}_i,\varepsilon_{i-1}),(\mathbf{k}_{i+1},\varepsilon_{i+1}),\ldots\right]\\ &\quad-icA_{n-1}^{\mathbf{P}}\left[\ldots,(\mathbf{k}_{i-1},\varepsilon_{i-1}),(\mathbf{k}_{i+1}+\mathbf{k}_i,\varepsilon_{i+1}),\ldots\right]\,.\end{aligned} \tag{2.8}$$

If $\mathbf{k}_i$ and $\mathbf{k}_{i+1}$ are not colinear, then $\mathbf{k}_i+\mathbf{k}_{i+1}$ is not null. As will become clear, the expression $A^{\mathbf{P}}$ does not vanish off-shell. Here we note that even if it would have been zero for off-shell gluon momenta, the right hand side of (2.8) would not be zero in the case were two adjacent gluons are colinear. In string theory such corrections are absent, a conclusion which follows from the *canceled propagator argument*. Analyticity implies that the amplitude with a longitudinal external state must vanish for all values of the $\mathbf{k}$'s. We will see below that the correct loop-sum formula for the scattering amplitude will have additional contributions which can restore gauge invariance (2.7) of the amplitude.

Note that for $n=2$, the two color orderings of the external gluons are the same. In that case s_1 and s_2 can be independently integrated over the whole loop, and therefore $A_2^{\mathbf{P}}=0$ whenever a polarization vector is longitudinal. The integral over the $\mathbf{x}$ zero-mode leads to a momentum conservation delta function. We therefore conclude that

$$A_2^{\mathbf{P}}[(\varepsilon_1,\mathbf{k}_1),(\varepsilon_2,\mathbf{k}_2)]\propto\delta(\mathbf{k}_1+\mathbf{k}_2)\;\varepsilon_1^\mu\varepsilon_2^\nu\left(k_\mu k_\nu-\mathbf{k}^2\eta_{\mu\nu}\right)G(\mathbf{k}^2)\,, \tag{2.9}$$

where G is some smooth function.

In the section 4, we will *derive* the correct analog of (2.2) for any planar gauge theory in a specific regularization inspired by the Alday-Maldacena $z_{\rm IR}$ regularization. We will interpret (2.9) as the scalar correction to the planar gluon propagator. Along the way we will see where and how $1/N$ corrections can be incorporated.

3. T duality and loop space Fourier transform

Let $\mathcal{C}_{\mathbf{x}}$ represent a closed loop and let $F[\mathcal{C}_x]$ be some functional on loop space. That is, if $\mathbf{x}(s)$ is some parametrization of $\mathcal{C}_{\mathbf{x}}$, then $F[\mathbf{x}(\cdot)]$ depends only on the image of $\mathbf{x}(s)$ and not on its specific parametrization. An example of such a functional is the expectation value of a Wilson loop operator in SU(N) gauge theory:

$$W\left[\mathcal{C}_{\mathbf{x}}\right] = \frac{1}{N}\langle \mathrm{tr} P e^{\oint \mathbf{A}\cdot d\mathbf{x}}\rangle\,, \tag{3.1}$$

where $\oint \mathbf{A}\cdot d\mathbf{x} = \int \mathbf{A}(\mathbf{x}(s))\cdot\dot{\mathbf{x}}(s)ds$.

Let $\mathcal{C}_{\mathbf{p}}$ be some closed loop in momentum space and let $\mathbf{p}(s)$, $s \in [0,1]$ be some parametrization of $\mathcal{C}_{\mathbf{p}}$. We define a Fourier transform of $F[\mathcal{C}_{\mathbf{x}}]$ along $\mathbf{p}(\cdot)$ to be

$$\widetilde{F}[\mathbf{p}(\cdot)] \equiv \int [\mathcal{D}\mathbf{x}] e^{i\oint \mathbf{p}\cdot d\mathbf{x}} F[\mathbf{x}(\cdot)] \tag{3.2}$$

and the corresponding *momentum Wilson loop* to be

$$\langle \widetilde{W}[\mathbf{p}(\cdot)]\rangle \equiv \int [\mathcal{D}\mathbf{x}] e^{i\oint \mathbf{p}\cdot d\mathbf{x}} \langle W[\mathbf{x}(\cdot)]\rangle\,, \tag{3.3}$$

where $\int[\mathcal{D}\mathbf{x}]$ is an integral over all closed curves. In (3.2) we have not specified the measure with respect to which the integral over closed curves is taken. Different such measures lead to different Fourier transforms, and in general the result depends of the specific parametrization of $\mathcal{C}_{\mathbf{p}}$, $\mathbf{p}(\cdot)$. Here we would like to mention one such measure considered by Migdal [9]. This is the measure where the worldline metric in $\int[\mathcal{D}\mathbf{x}]$ is the induced metric from the map $\mathbf{p}(s)$ to momentum space. This measure is special because the result depends only on $\mathcal{C}_{\mathbf{p}}$ and not on its specific parametrization.[1] Nevertheless, this is not the measure we will derive from the scattering amplitude; what we will find is closer to what we would write by analogy with string theory.

Note that for any point in the integral over the gluon insertion points $\{s_i\}$, the momentum dependence of $A_n^{\mathbf{P}}$ can be written as

$$\sum \mathbf{k}_i\cdot\mathbf{x}(s_i) = \int ds \sum \mathbf{k}_i\cdot\mathbf{x}(s)\delta(s-s_i) = -\int_0^T ds\ \mathbf{p}\left(\frac{s}{T}\right)\cdot\dot{\mathbf{x}}(s)\,,$$

where $\mathbf{p}(s)$ is the polygon momentum loop

$$\mathbf{p}(s) = \sum_i \mathbf{k}_i\theta(s - s_i/T)\ . \tag{3.4}$$

The heuristic scattering formula (2.2) can therefore be written in terms of the momentum loop (3.4):

$$A_n^{\mathbf{P}} = Z_3^{-n}\prod_i \int_{s_{i-1}}^1 ds_i\varepsilon_i^{\mu}\frac{\delta}{\delta p^{\mu}(s_i)}\langle\widetilde{W}[\mathbf{p}(\cdot)]\rangle\,, \tag{3.5}$$

[1] Of course replacing the measure $\int[\mathcal{D}\mathbf{x}]_{\text{Migdal}}$ by $\int[\mathcal{D}\mathbf{x}]_{\text{Migdal}}G[\mathcal{C}_x]$ for some reparametrization-invariant functional G will lead to another reparametrization-invariant Fourier transform.

where

$$\langle\widetilde{W}[\mathbf{p}(\cdot)]\rangle = \int_0^\infty \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}]_1\, e^{-\frac{1}{2}\int_0^T \dot{\mathbf{x}}^2 ds}\langle W[\mathbf{x}(\cdot)]\rangle e^{-i\oint \mathbf{p}\cdot d\mathbf{x}}$$

is a momentum Wilson loop, defined with respect to the measure for loops discussed in the previous section. This Fourier transform is, however, not reparametrization invariant. A reparametrization invariant result is obtained only after dressing $\langle\widetilde{W}[\mathbf{p}(\cdot)]\rangle$ into $A_n^{\mathbf{P}}$ (3.5). This dressing (3.5) can be thought of as an integration over all reparametrizations of the polygon $\mathbf{p}(s)$ (3.4).

One of the properties of momentum-space Wilson loops is that $\mathbf{p}(s)$ can have discontinuities. In the study of gluon scattering amplitudes, such discontinuities naturally appear as the T-dual of the gluon momenta.

In [2] Alday and Maldacena gave a prescription for the holographic computation of gluon scattering amplitudes using open strings in AdS. The prescription amounts to computing an open string amplitude on a probe brane in AdS. At large 't Hooft coupling, the open string amplitude is approximated by its saddle point. In that approximation, one can neglect the polarization dependence of the scattered gluons. The open string amplitude is expressed through an integral over the moduli space of the insertion points of the gluon vertex operators. In order to find the saddle point, Alday and Maldacena first did a T-duality along the 3+1 transverse directions (see [16] for details). That T-duality commutes with the integral over the insertion points and can be done point by point.

Next, we show that after suppressing the polarization dependence of the gluon vertex insertions, this T-duality amounts to a Fourier transform in loop space. More generally, T-duality takes the form of a Fourier transform in loop space whenever one studies an open string that ends on some closed curve in a specific parametrization. A meaningful result may then be obtained by summing over all possible reparametrizations of the curve or by a saddle point approximation to that sum [2].

To see this, consider the string dual of the Wilson loop (3.1). That is, consider an open string in AdS_5 with boundary conditions such that it ends on the curve $\mathbf{x}_b(s)$ at the boundary of AdS (at $z_{\mathrm{UV}} \to 0$). In Poincaré AdS

$$ds^2 = R_{\mathrm{AdS}}^2 \frac{dz^2 + dx_{3+1}^2}{z^2}\,, \tag{3.6}$$

the string worldsheet action in conformal gauge is

$$S_0 = \frac{\sqrt{\lambda}}{4\pi}\int_{\mathcal{D}} d\sigma\partial\tau\ \left[(\partial_\alpha \mathbf{x})^2 + (\partial_\alpha z)^2\right]/z^2\ . \tag{3.7}$$

The Wilson loop expectation value is roughly (ignoring the ghosts and other worldsheet fields for the moment)

$$\langle W\left[\mathcal{C}_{\mathbf{x}}\right]\rangle = \int [Df(\sigma)]\int [D\mathbf{x}(\sigma,\tau)]_{\mathbf{x}(\sigma,0)=\mathbf{x}_b(f(\sigma))}\, D\left[z(\sigma,\tau)\right]_{z(\sigma,0)=0}\, e^{-S_0}\,, \tag{3.8}$$

where the f-integral is over the group of boundary reparametrizations [17], and $\mathbf{x}_b(\sigma)$ is some non-degenerate parametrization of $\mathcal{C}_{\mathbf{x}}$. Next, we do a change of variables in the path integral which can be described as a "T-duality" along the non-compact 3+1 flat directions. To do this, we follow Buscher [18]. For each field x^μ, we gauge the shift symmetry $x^\mu \to x^\mu + \lambda^\mu$, and introduce a worldsheet gauge field V^μ_α and a scalar lagrange multiplier p^μ. We then consider the gauge-invariant action

$$S_1 = \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma \partial\tau \ \left[(\partial_\alpha \mathbf{x} - \mathbf{V}_\alpha)^2 / z^2 - i \mathbf{p} \cdot \mathbf{F} \right] , \tag{3.9}$$

where $\mathbf{F} = \partial_\tau \mathbf{V}_\sigma - \partial_\sigma \mathbf{V}_\tau$ and we are suppressing the kinetic term for z. Next, we fix a gauge by absorbing $d\mathbf{x}$ into the gauge field. The resulting gauge field $\mathbf{V}$ is subject to the boundary condition $\mathbf{V}_\sigma(\sigma,0) = \partial_\sigma \mathbf{x}_b(\sigma)$. To see this, introduce a boundary auxiliary field $\mathbf{b}(\sigma)$ and rewrite the Dirichlet boundary conditions for $\mathbf{x}$ by adding to the boundary action the term

$$i \int d\sigma \mathbf{b}(\sigma) \cdot [\mathbf{x}(\sigma,0) - \mathbf{x}_b(\sigma)] \ . \tag{3.10}$$

Now when we gauge the shift symmetry of $\mathbf{x}$, (3.10) is replaced by

$$S_2 = i \int d\sigma \mathbf{b}(\sigma) \cdot \left[[\mathbf{x}(\sigma,0) - \mathbf{x}(0,0)] - \left[\int_0^\sigma ds \mathbf{V}_\sigma(s) - \mathbf{x}(0,0) \right] - \mathbf{x}_b(\sigma) \right] \ . \tag{3.11}$$

After we absorb $d\mathbf{x}$ into the gauge field, the term $[\mathbf{x}(\sigma,0) - \mathbf{x}(0,0)]$ is removed from (3.11). Integrating over $\mathbf{b}(\sigma)$ yields

$$\delta\left(V_\sigma - \partial_\sigma \mathbf{x}_b\right) \delta\left(\mathbf{x}(0,0) - \mathbf{x}_b(0)\right) \ . \tag{3.12}$$

If we first integrate out $\mathbf{p}$, then $\mathbf{V}$ becomes a flat connection $V = d\tilde{\mathbf{x}}$. By defining $\tilde{\mathbf{x}}(0,0) = \mathbf{x}(0,0) = \mathbf{x}_b(0)$, we see that (3.9) is equivalent to the original action. If on the other hand, we first integrate $\mathbf{V}$, then it is convenient to integrate by parts in the second term in (3.9). We then have

$$\begin{aligned} S_1 + S_2 = & \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma \partial\tau \ \left[(\mathbf{V}_\alpha \cdot \mathbf{V}_\alpha)/z^2 + i\left(\mathbf{V}_\sigma \cdot \partial_\tau \mathbf{p} - \mathbf{V}_\tau \cdot \partial_\sigma \mathbf{p} \right) \right] - i \frac{\sqrt{\lambda}}{4\pi} \int_{\partial\mathcal{D}} d\sigma \mathbf{V}_\sigma \cdot \mathbf{p} \\ & - i \int_{\partial D} d\sigma \mathbf{b}(\sigma) \cdot \left[\int_0^\sigma ds \mathbf{V}_\sigma(s) - \mathbf{x}(0,0) + \mathbf{x}_b(\sigma) \right] \ . \end{aligned} \tag{3.13}$$

Now, by integrating out $\mathbf{V}$ we get

$$\mathbf{V}_\sigma = -iz^2 \partial_\tau \mathbf{p} \, , \qquad \mathbf{V}_\tau = iz^2 \partial_\sigma \mathbf{p} \, , \qquad \mathbf{b}(\sigma) = \frac{\sqrt{\lambda}}{4\pi} \partial_\sigma \mathbf{p} \ . \tag{3.14}$$

By plugging (3.14) back into (3.13) the action for $\mathbf{p}$ becomes

$$\begin{aligned} S_3 &= \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma \partial\tau \ \left[z^2 (\partial_\alpha \mathbf{p})^2 + (\partial_\alpha z)^2 / z^2 \right] - i \frac{\sqrt{\lambda}}{4\pi} \int_{\partial\mathcal{D}} d\sigma \partial_\sigma \mathbf{p} \cdot [\mathbf{x}_b - \mathbf{x}(0,0)] \\ &= \frac{\sqrt{\lambda}}{4\pi} \int_{\mathcal{D}} d\sigma \partial\tau \ \left[z^2 (\partial_\alpha \mathbf{p})^2 + (\partial_\alpha z)^2 / z^2 \right] - i \frac{\sqrt{\lambda}}{4\pi} \int_{\partial\mathcal{D}} d\sigma \partial_\sigma \mathbf{p} \cdot \mathbf{x}_b \ . \end{aligned} \tag{3.15}$$

If we now rescale $\mathbf{p}$ and z as $(\mathbf{p}, z) \rightarrow \left(-\frac{4\pi}{\sqrt{\lambda}}\mathbf{p}, \frac{\sqrt{\lambda}}{4\pi}z\right)$, then (3.3) is reproduced if we identify the T-dual Wilson loop with an open string in dual AdS that ends on the Poincaré horizon. The dual AdS metric is

$$ds^2 = R_{\text{AdS}}^2 \frac{dr^2 + dp_{3+1}^2}{r^2}\,, \tag{3.16}$$

where the radial direction is $r = \frac{1}{z}$.[2] The path integral over $\mathbf{p}(\sigma,\tau)$ splits into an integral over its boundary value $\mathbf{p}(\sigma,0) = \mathbf{p}_b(\sigma)$ and an integral over its bulk value $\mathbf{p}(\sigma,\tau>0)$, subject to the Dirichlet boundary condition, $\mathbf{p}(\sigma,\tau=0) = \mathbf{p}_b(\sigma)$.

To summarize, we have shown that

$$\begin{aligned}\int [Dz]_{z_\partial=0}\,[D\mathbf{x}]_{\mathbf{x}_\partial=\mathbf{x}_b}\,e^{-S_0[\mathbf{x},z]} &= \int [Dz]_{z_\partial=0}\,[D\mathbf{x}]\,[D\mathbf{b}]\,[D\mathbf{V}]\,[D\mathbf{p}]\,e^{-S_1-S_2}\\ &= \int [Dz]_{z_\partial=0}\,[D\mathbf{p}]\,e^{-S_3[\mathbf{x}_b,z,\mathbf{p}]} = \int [Dz]_{z_\partial=0}\,[D\mathbf{p}]\,e^{-S_0[\mathbf{p},1/z]+i\oint \mathbf{x}_b\cdot d\mathbf{p}}\\ &= \int [D\mathbf{p}_b(s)]\,e^{i\oint \mathbf{x}_b\cdot d\mathbf{p}_b}\int [Dz]_{z_\partial=0}\,[D\mathbf{p}]_{\mathbf{p}_\partial=\mathbf{p}_b}\,e^{-S_0[\mathbf{p},1/z]}\,.\end{aligned} \tag{3.17}$$

If we try to start in the opposite direction by considering an open string in AdS_5 with boundary conditions such that it ends on the curve $\mathcal{C}_{\mathbf{x}_b}$ at the Poincaré horizon $z=\infty$ then the term

$$\int_{\partial\mathcal{D}} d\sigma z^2 \partial_\tau \mathbf{p}\cdot\mathbf{p} \tag{3.18}$$

(which is now at $z=\infty$) is not zero (and diverges due to the $z^2 \to \infty$ factor). However, that term exactly cancels between $\int_{\partial\mathcal{D}} d\sigma \mathbf{V}_\sigma \cdot \mathbf{p}$ and $\int_{\partial D} d\sigma \mathbf{b}(\sigma)\int_0^\sigma ds \mathbf{V}_\sigma(s)$ in (3.13).

We have therefore seen, by using the identification of the Wilson loop operator with open strings ending on the boundary of AdS, that T-duality on the string worldsheet reproduces the Fourier transform in loop space (3.3). As noted in [2], the T-dual AdS space where the worldsheet fields are $\mathbf{p}$ and r, has a non trivial dilaton $\Phi \sim \log(r)$. Therefore, expectation values of Wilson loops in momentum space are dual to open strings in AdS ending on the Poincare horizon, where a dilaton is turned on in the bulk, but no NS B field is needed.[3]

4. Scattering amplitudes from worldline path integrals

In this section we will derive a general relation between scattering amplitudes of adjoint fields in SU(N) gauge theories and Wilson loops.[4] The derivation will follow an example (gluon scattering in pure YM theory), but holds for the scattering of any adjoint fields in the planar limit.

Unless some couplings are scaled with N, fields that are not in the adjoint representation do not contribute to planar scattering amplitude of adjoint fields. Therefore, for our purpose, we can truncate the theory to the fields in the adjoint only.

[2] Note that if we change variables in the path integral from z to $r = \frac{1}{z}$, then we generate a dilaton $\Phi \sim \log(r)$.

[3] This is a reference to a suggestion of [5].

[4] Our derivation applies for any theory with an 't Hooft limit.

Let $\varphi_I^b(\mathbf{k})$ be the set of all fields in the adjoint representation. These are the fields we wish to scatter. The index b is an SU(N) adjoint color index, the index I denotes the field together with all of its labels, such as spin, flavor or other global charge; $\mathbf{k}$ is the momentum. Along with the general discussion, we will follow the example where the only field in the adjoint is the gauge field. In that case $I = \mu$ is a vector index.

We will want to include a source for φ, so that we may scatter external φ "particles"; with this in mind, consider the generating function

$$Z[J] \equiv \langle e^{\int J\cdot\varphi} \rangle \,,$$

where $J \cdot \varphi = J_b^I \varphi_I^b$. Note that if φ is fermionic, so is J.

To study on-shell scattering of φ particles, we will use the LSZ formula

$$\mathcal{A}_n\left[(\mathbf{k}_1, \varepsilon_{b_1}^{I_1}), \ldots, (\mathbf{k}_n, \varepsilon_{b_n}^{I_n})\right] = \prod_{i=1}^{n} \left(\lim_{\mathbf{k}_i^2 \to m^2} \varepsilon_{b_i}^{I_i} \left(G_{\mathrm{ph}}^{-1}(\mathbf{k}_i) \right)_{I_i}^{K_i} \frac{\delta}{\delta J_{b_i}^{K_i}(\mathbf{k}_i)} \right) Z[J]|_{J=0} \,, \tag{4.1}$$

where G_{ph}^{-1} is the fully-dressed propagator, m is the physical mass (so G_{ph} has a pole at $\mathbf{k}^2 = m^2$), and we work in the convention where all momenta are out-going.

For our YM example, in Feynman gauge

$$\lim_{\mathbf{k}^2 \to m^2} \left(G_{\mathrm{ph}}^{-1}(\mathbf{k}) \right)_{\mu}^{\nu} = \lim_{\mathbf{k}^2 \to 0} h(\lambda)\; \eta_{\mu}^{\nu}\; \mathbf{k}^2 \,,$$

where $h(\lambda)$ is some function of the 't Hooft coupling λ, and we used the fact that the Ward identity protects the location of the pole of G_{ph}.

In the planar limit, a fixed color index is attached to any piece of a planar diagram boundary between two adjacent insertions. As a result, for fixed number of external colored fields (n), we can split the N color indices as $N = n + M$, where $N, M \to \infty$ with n fixed. The SU(N) gauge group then naturally splits into SU(n) $\times$ SU(M), where the amplitude transforms covariantly under the global SU(n) symmetry and is invariant the global SU(M) symmetry (as well as the full SU(N) local symmetry).

We split the fields accordingly as

$$\varphi = (a, w, A) \,,$$

where a are the fields that transform in the adjoint of SU(n),[5] w are the fields that transform in the bi-fundamental of SU(n) $\times$ SU(M) and A are the fields transforming in the adjoint of SU(M). The source J is then coupled only to a. Note that this splitting of the fields is unambiguously defined only asymptotically (where we set the states being scattered) and is gauge-dependent in the bulk. Therefore, it requires partial gauge fixing (as will be done for our example in section 6), and some of the w fields are the corresponding ghosts. This gauge-fixing leaves an SU(n) $\times$ SU(M) $\subset$ SU(N) subgroup unfixed.

At large N, the a field contributes only at tree level. Any w field goes only on the boundaries of planar diagrams and therefore contributes only at one loop to one-particle

[5] We will reserve the non-boldface a for the generic SU(n) adjoint, which are to be distinguished from the $\mathbf{a}$s which will specifically represent the gauge fields in the YM example of section 6.

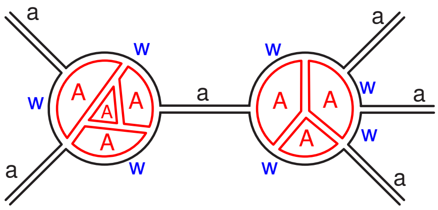

Figure 1: A generic (reducible) planar diagram contributing to the scattering amplitude of five a fields.

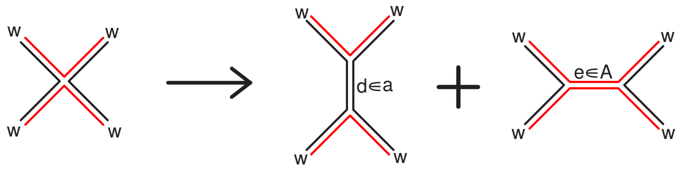

Figure 2: An auxiliary field is added to open up a four-w vertex.

irreducible (1PI) diagrams. We will incorporate these simplifications by first computing all planar 1PI diagrams bounded by w and then connecting these into trees with a propagators (see figure 1).

To do so, we first add auxiliary fields transforming in the adjoint of SU(n) to express all interactions involving four w's (and in general higher) as interactions of only two w's with the auxiliary fields. For instance, in our pure YM example there is a non-abelian interaction involving four w-bosons. As will be explained in section 6, we first express it as (see figure 2)

$$\begin{aligned} \mathrm{tr}_{n\times n}\left(w_{[\mu},w^\dagger_{\nu]}w^{[\mu}w^{\nu]\dagger}\right) &\quad\rightarrow\quad 2\mathrm{tr}_{n\times n}\left(w_{[\mu}w^\dagger_{\nu]}d^{\mu\nu}\right)-\mathrm{tr}_{n\times n}\left(d^{\mu\nu}d_{\mu\nu}\right) \\ \mathrm{tr}_{M\times M}\left(w^\dagger_{[\mu},w_{\nu]}w^{[\mu\dagger}w^{\nu]}\right) &\quad\rightarrow\quad 2\mathrm{tr}_{M\times M}\left(w^\dagger_{[\mu}w_{\nu]}e^{\mu\nu}\right)-\mathrm{tr}_{M\times M}\left(e^{\mu\nu}e_{\mu\nu}\right), \end{aligned} \tag{4.2}$$

where $w_\mu = w^b_\mu T^b_{n\times M}$ and $d\in a,\ e\in A$ are anti-symmetric auxiliary tensor fields. The arrows indicate classical equivalence, which is exact at large N (for fixed number of external particles).[6] The resulting action is now quadratic in w. Integrating it out results in a one-loop determinant, and the generating function is now

$$Z[J]=\langle\det\left(\frac{\delta^2 S}{\delta^2 w}\right)_{w=0} e^{\int J\cdot a}\rangle_{a,A}\ . \tag{4.3}$$

As will be derived in section 6, for our pure YM example in Feynman gauge

$$\det\left(\frac{\delta^2 S}{\delta^2 w}\right)_{w=0} = \det\left[-D^2\right]\det\left[-D^2\eta+i2F-d-e\right]^{-\frac{1}{2}}\ . \tag{4.4}$$

[6]Instead of introducing these two types of auxiliary fields, we could express the two vertices in (4.2) with a single auxiliary field in the adjoint of SU(n) only ($d^{\mu\nu}\in a$). Such an auxiliary tensor field would not be anti-symmetric. Both types of auxiliary fields have advantages and disadvantages that will be discussed in section 6. Here, for simplicity, we have chosen the first option.

If the scattered partons are massless, then there are IR divergences which should be regularized. In section 6 we add a Higgs field which gives the w-boson a mass, in order to regulate these IR divergences in the gluon amplitude. Our formal derivation here will rely only on large N, and for massless partons different IR regulators may be used.

Next, we express the one loop determinant using the worldline formalism. That should be possible at least for any gauge theory that can be obtained as the low energy limit of some open string theory. In such a string theory description, the adjoint fields are represented by open strings stretched between D-branes. The field theory one loop determinant is then obtained from the string theory as the low energy limit of the annulus diagram in which the string length goes to zero. In that limit, the annulus becomes a circle and the string worldsheet theory becomes a worldline theory on the circle. The one loop determinant in (4.3) is coupled to arbitrary background fields. When coupling an open string to a background field, the worldsheet conformal symmetry restricts the background to be on-shell. In the worldline limit however, there is no two dimensional conformal symmetry and one can consistently take the background off shell.[7]

As we will review in the next section and the appendices, the worldline representation of the one loop determinant is of the following form

$$\det\left(\frac{\delta^2 S}{\delta^2 w}\right)_{w=0} = \exp\left(\int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}]_1 [D\xi] \mathrm{tr} P e^{iS_{\mathrm{wl}}[\mathbf{x},\xi;a,A]}\right),$$

where $\mathbf{x}(s)$ is the worldline path, ξ stands for all other worldline fields, S_{wl} is the worldline action and P stands for path ordering. Since the derivatives in the one loop determinant are covariant derivatives, the worldline action will contain a linear coupling to the gauge field

$$\oint (\mathbf{A}-\mathbf{a})\cdot d\mathbf{x} \ \in \ S_{\mathrm{wl}}\,,$$

where the coupling of $\mathbf{x}$ to $\mathbf{a}$ and $\mathbf{A}$ have opposite signs, since $\mathbf{w}$ is in the $(\mathbf{n},\bar{\mathbf{M}})$ representation. As such we will call the path ordered exponent of the worldline action a *generalized Wilson loop* and the worldline path integral a *Wilson sum*.

The worldline representation of the YM example (4.4) is given in section 6 and the appendices, following [15]. In that description we have a sum over two worldline path integrals, one representing the vector determinant and the other representing the ghost determinant (which therefore has a minus sign in front). That worldline representation may be considered a gauged-fixed version of a single worldline path integral with some local (super) symmetries, where the ghost determinant piece comes from some worldline ghosts. The reason it broke up into a sum of two worldline path integrals is that these are the one-particle worldline theories standing in the exponent and we don't have background ghost, as in tractable string worldsheet theories. That is because (4.4) will be obtained without fixing a gauge for the $\mathrm{SU}(n)\times\mathrm{SU}(M)$ part of the gauge group. As we will scatter

[7]For the Brink-Schwarz and pure spinor worldline descriptions of 10d $\mathcal{N}=1$ or 4d $\mathcal{N}=4$ SYM theories, the background is restricted to be on-shell. This is, however, a technical complication resulting from the absence of an off-shell superspace description of these gauge theories. It may be resolved by a harmonic superspace worldline description of these theories [19].

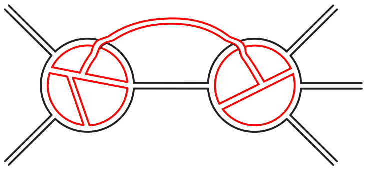

Figure 3: A class of non-planar diagrams that are down by $\frac{1}{M}$ and are removed in (4.6).

the SU(n) fields, we will fix a gauge for that part as well and add the corresponding ghost as part of the a fields. However, since we do not scatter these SU(n) ghosts (which, by large-N, contribute only at tree level), ghost number conservation implies they will not contribute in the planar limit and will be harmlessly set to zero.

Next, we split the gauge theory path integral into an explicit path integral over a (from which only the tree level contributes at large N) and a path integral over A written in terms of expectation values $\langle \ldots \rangle_A$. The resulting generating functional of correlation functions of as is

$$Z[J] = \int [Da] e^{\int (S[a] + J\cdot a)} \left\langle \exp\left(\int \frac{dT}{T} \mathcal{N} \int [D\mathbf{x}][D\xi] \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]} \right) \right\rangle_A , \tag{4.5}$$

where $S[a]$ is the SU(n) gauge theory action (which includes the auxiliary field couplings in (4.2)). In (4.5) and in the remainder of this section, by $[D\mathbf{x}]$ we mean the measure $[D\mathbf{x}]_1$, defined in (2.4).

For our example, in Feynman gauge, the piece in (4.5) outside of the $\langle \ldots \rangle_A$ expectation value is

$$\int [Da]\; e^{\int (S[a]+J\cdot a)} \cdots = \int [D\mathbf{a}][Dc][Dd] e^{\int \mathrm{tr}\left(\frac{1}{2}\mathbf{a}\cdot\Box\mathbf{a} + \bar{c}\Box c + \mathbf{J}\cdot\mathbf{a} - d^2/4\right)} \cdots \quad .$$

Up to this point, we have not used the large N limit and our expression for the generating function (4.6) is exact for any value of N. We now use large N factorization of Wilson loop expectation values to lift the A-expectation value into the exponent

$$Z[J] = \int [Da] \exp\left(\int (S[a] + J\cdot a) + \int \frac{dT}{T} \mathcal{N} \int [D\mathbf{x}][D\xi] \langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]} \rangle_A \right) . \tag{4.6}$$

In writing equation (4.6) we have removed a class of (non-planar) diagrams that are down by $\frac{1}{M}$ and are shown in figure 3. There are also diagrams contributing to (4.6) that are down by powers of $\frac{n}{M}$ (planar and non-planar) (see figure 4). In a diagrammatic expansion of the path integral over a, the generalized Wilson loop expectation values

$$\langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]} \rangle_A$$

play the role of vertices of any order (which are not local in space). In that diagrammatic expansion, we need to keep only the tree level contributions. This amounts to keeping only the leading contribution in $\frac{n}{M}$ since the color index in an a loop runs only over n indices

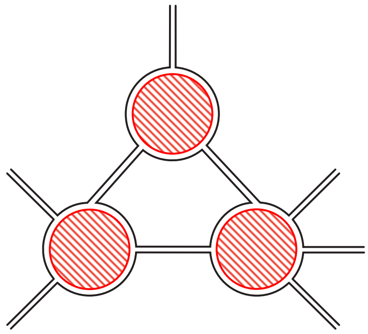

Figure 4: A class of planar diagrams that are down by $\frac{n}{M}$ and are removed by keeping only the tree level contributions in a.

(see figure 4). In other words, the planar 1PI *quantum effective action* of the a fields is[8,9,10]

$$\Gamma[a] = S[a] + \int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}][D\xi]\langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]}\rangle_A \tag{4.7}$$

and the sum over connected diagrams with sources is

$$W[J] = S[a_J] + \int J\cdot a_J + \int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}][D\xi]\langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a_J,A]}\rangle_A\,,$$

where

$$-J = \frac{\delta}{\delta a}\left(S[a] + \int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}][D\xi]\langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]}\rangle_A\right)_{a=a_J}\,.$$

In particular, for two separated points $\mathbf{y}\neq\mathbf{z}$, the fully-dressed inverse propagator is given by

$$G_{ph}^{-1}(\mathbf{y},\mathbf{z})_{b_1b_2} = \frac{\delta^2}{\delta a(\mathbf{y})\delta a(\mathbf{z})}\left[S[a] + \int \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}][D\xi]\langle \mathrm{tr}\left(P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]}\right)\rangle_A\right]_{a=0}\delta_{b_1b_2}\,. \tag{4.8}$$

When an external leg is dressed by G_{ph} (4.8), we get a pole which is then canceled by G_{ph}^{-1} in (4.1). The 1PI m-vertices V_m with $m\geq 5$ are given by

$$\begin{aligned}
&V_m\left[(\mathbf{k}_1,\varepsilon_{b_1}^{I_1}),\ldots,(\mathbf{k}_m,\varepsilon_{b_m}^{I_m})\right]\\
&=\prod_{i=1}^{m}\int d\mathbf{x}_i e^{-i\mathbf{k}_i\cdot\mathbf{x}_i}\varepsilon_{b_i}^{I_i}\frac{\delta}{\delta a_{I_i}^{b_i}(\mathbf{x}_i)}\int\frac{dT}{T}\mathcal{N}\int[D\mathbf{x}][D\xi]\langle\mathrm{tr}_{N\times N}\left(P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;a,A]}\right)_{a=0}\rangle_A\\
&=\sum_{\pi}\mathrm{tr}_{n\times n}\left(T^{b_{\pi(1)}}\ldots T^{b_{\pi(m)}}\right)\int\frac{dT}{T}\mathcal{N}\int[D\mathbf{x}][D\xi]\prod_{i=1}^{m}\int_{s_{\pi(i-1)}}^{s_{\pi(i+1)}}ds_{\pi(i)}\varepsilon_{b_i}^{I_i}V(s_1)_{I_i}e^{-i\mathbf{k}_i\cdot\mathbf{x}(s_i)}\\
&\quad\times\langle\mathrm{tr}_{M\times M}\left(P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;0,A]}\right)\rangle_A\,,
\end{aligned}\tag{4.9}$$

[8]Note that the last (worldline) term can also contribute to the mass of a, in the case that it is not protected by a Ward identity.

[9]The structure of the non-planar corrections is very interesting but we postpone their discussion.

[10]A formula of this form, incorporating only the contributions of scalars and fermions in w (but not w-bosons), appears in section 6 of [8].

where the sum is over all permutations, $s_{\pi(0)} = 0$, $s_{\pi(m+1)} = T$ and

$$\varepsilon_b^I T^b V_I(s) = -\frac{\delta}{\delta a_I^b(\mathbf{x}(s))} S_{\rm wl}[\mathbf{x}, \xi; a, A]_{a=0}$$

is the worldline vertex operator.[11] For $n \leq 4$, in addition to (4.9) one also have to include the a-tree level vertex. In our example when $a = a_b^\mu$ is the gauge field, we have

$$\varepsilon_b^\mu V_\mu(s) T^b = \varepsilon_b^\mu \dot{x}_\mu(s) T^b$$

from the scalar worldline and

$$\varepsilon_b^\mu V_\mu(s) T^b = \varepsilon_b^\mu \left(\dot{x}_\mu(s) + i k_i^\nu [\psi_\nu, \bar{\psi}^\mu]\right) T^b$$

from the vector worldline, where ψ^μ is a complex worldline fermion field (see section 6).

In (4.9) we assumed that $S_{\rm wl}$ is linear in a. However, in our example in section 6, $S_{\rm wl}$ will have quadratic coupling to a. In our example, the vector worldline contains the non-abelian coupling

$$S_{\rm wl} = \int ds \psi^\mu \bar{\psi}^\nu F_{\mu\nu} + \cdots = \int ds \psi^\mu \bar{\psi}^\nu [a_\mu, a_\nu] + \dots \ . \qquad (4.10)$$

This term in $S_{\rm wl}$ is necessary for the gauge invariance of the operator

$$\int [D\mathbf{x}][D\xi] {\rm tr} P e^{-S_{\rm wl}[\mathbf{x}, \xi; a, A]} \ .$$

It leads to what is called the *two gluon vertex* in the formulation of the theory in terms of worldline path integrals [15]. In general it reads

$$\varepsilon_b^I \varepsilon_c^J T^b T^c V_{IJ}(s) = \frac{\delta}{\delta a_I^b(\mathbf{x}(s))} \frac{\delta}{\delta a_J^c(\mathbf{x}(s))} S_{\rm wl}[\mathbf{x}, \xi; a, A]_{a=0} \ .$$

The corresponding vertex is indicated in figure 5b. These contribute only at the boundary of the integration over the vertices insertion points (see figure 5a).[12] In our example, by partial integration, the coupling to the field strength $F_{\mu\nu}$ can be replaced by [9, 21, 22]

$$\int ds \psi^\mu \bar{\psi}^\nu F_{\mu\nu} \ \rightarrow \ \int ds \psi^\mu \bar{\psi}^\nu \dot{x}_{[\mu} \ddot{x}_{\nu]} \ . \qquad (4.11)$$

Using this 'radiation-reaction term,' the two-gluon vertex can be avoided.

Similarly, since the auxiliary fields couple only linearly to the worldline action, are never external, and have no kinetic terms, integrating them out leads to an interaction between position space Wilson loops only when these touch at a point. As can be seen in figure 6, these are in one-to-one correspondence with points in the path integral over closed loops where the loop is self-crossing. Therefore, these contributions can be represented by a correction to self-crossing points of generalized Wilson loops. We can represent these corrections by replacing the generalized Wilson loop expectation values with

[11] Note that in case where the loop has self-crossings, the variation with respect to the source at the crossing point will give a sum of the two possible orderings.

[12] In a superspace representation of the worldline path integral, the path ordered exponent is replaced by a super-path-ordered exponent (see [20] for an $\mathcal{N} = 1$ example). In that representation of the worldline 1PI vertex there are no two-gluon vertices and the corresponding boundary contact terms result from the super-path-ordering.

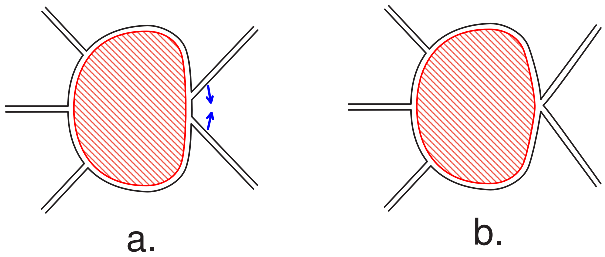

Figure 5: a. A boundary point in loop space where a sub-loop between two adjacent insertions collapses to a point. **b.** A w-loop interacting with two a fields at a four-vertex.

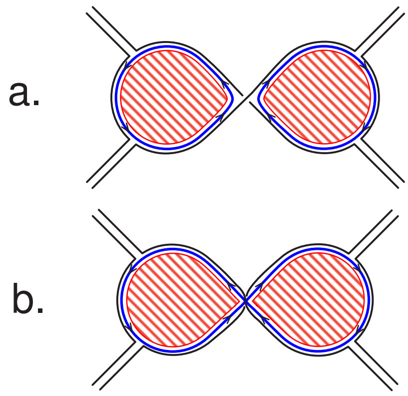

Figure 6: a. Two Wilson loops representing two w-loops interacting at a four-vertex. **b.** A Wilson loop with a crossing point representing a single w-loop.

$$\langle \mathrm{tr} P e^{-S_{\mathrm{wl}}[\mathbf{x},\xi;A]} \rangle_A^{s.c.} \,,$$

where $s.c.$ stands for *self-crossing correction*. Note that these corrections are localized on the loop and contribute at λ^1 only. Therefore, they can be computed perturbatively. We leave such computation to future work.

The momentum dependence of any 1PI m-vertex V_m is

$$e^{-i\sum_{i=1}^{m}\mathbf{k}_i\cdot\mathbf{x}(s_i)} = e^{i\oint \mathbf{p}\cdot d\mathbf{x}} \,,$$

where

$$\mathbf{p}(s) = \sum_{i=1}^{m} \mathbf{k}_i \theta(s - s_i) \ . \tag{4.12}$$

Therefore, each planar 1PI m-vertex is a *generalized momentum Wilson loop.*[13] In momentum loop space, the large N factorization of the partial amplitude into 1PI planar gluonic blobs connected by (dressed) propagators has a geometric manifestation as a sum

[13] Note that in the expression for the 1PI vertex, it is only after we integrate over the vertices insertion points that we find a meaningful, parametrization-invariant functional of the gluon momenta and polarizations.

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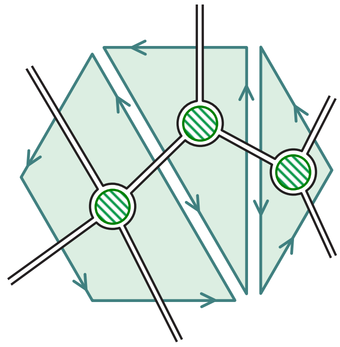

Figure 7: A one-particle-reducible contribution to the scattering amplitude is a subdivision of the polygonal momentum wilson loop.

over all the ways of subdividing the polygon made of the external momentums into sub-polygons connected by the a propagators (see figure 7).[14]

The expressions for the 1PI planar vertices and the propagator in terms of generalized Wilson loop expectation values represent a non-perturbative definition of these quantities. For the scattering amplitude of any finite number of partons (n), there is a finite number of tree level diagrams connecting the external legs. For 't Hooft coupling of order one, all of them contribute and should be summed over. The large 't Hooft coupling limit will be discussed in the next section. In that limit we will argue that for generic external momenta, the single 1PI n-vertex will dominate the sum. In momentum space it is given by a single polygon made of the ordered external momenta.

5. One-particle reducible contributions

In perturbation theory, there are non-1PI (one-particle reducible) contributions that are not suppressed by a power of $1/N$. Here, we wish to understand what is known about the contribution of this class of diagrams outside of perturbation theory.

In string theory, the moduli space of the disk with n insertion has boundaries. The components of this boundary are in one-to-one correspondence with the non-1PI diagrams. A separation between 1PI and non-1PI contributions is not gauge invariant. Similarly, in the string theory, restricting the integral over gluon vertex operators insertion points to some specific point away from the saddle point is not gauge invariant and depends on the specific worldsheet formalism being used. In any given worldsheet formalism, we suggest to identify the contribution to the string theory amplitude from the boundary of moduli space with the non-1PI contributions to the gauge theory amplitude in some specific gauge corresponding to the given worldsheet formalism. We will then use that identification to argue that at strong coupling, in any gauge, the 1PI contributions dominate the amplitude.

Alday and Maldacena [2] argue that the gluon scattering amplitude at large λ is dominated by a saddle point which for generic external momenta sits away from the boundary

[14] A very similar picture was used in [23] when realizing the dual conformal symmetry in $\mathcal{N} = 4$ perturbation theory.

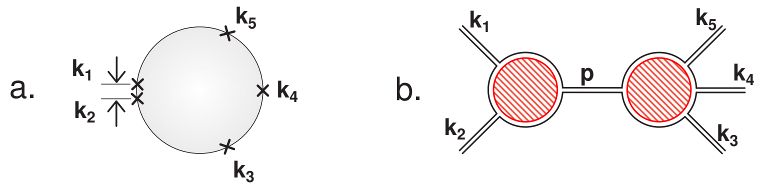

Figure 8: A one-particle reducible (i.e. non-1PI) diagram (b) and the corresponding boundary point in the disk moduli space (a).

of the moduli space. Therefore, assuming the identification above, the leading contribution to the amplitude cannot come from one-particle reducible diagrams at strong coupling.

In addition, if we restrict the integral to one of the boundary components (see figure 8), and look for the saddle point using the AM holographic prescription we find a contribution which is indeed much smaller than that of the leading saddle point. This had to be true given the previous statement, since we are here extremizing over a subset of the original set; unless the extremum of the bigger integral lies in the subset, the inequality will be strict.

We would like to use this fact to argue that the 1PI contributions dominate at large 't Hooft coupling. Specifically, the non-1PI contributions to our formula are expressed as a product of amplitudes each with one off-shell external leg, attached by a dressed gluon propagator with momentum $p \equiv k_1 + \cdots + k_r$. At large λ, we can try to compute each factor by a worldsheet saddle-point calculation. For generic $\{k_1, \ldots, k_r\}$, the area of the resulting worldsheet gets a divergent contribution from the region where it attaches to this non-null edge at the boundary. Heuristically, this is the tension of the flux tube carried by the off-shell gluon through the strongly-interacting theory.[15]

For special values of the external momenta, the saddle point of the big integral *does* lie on the boundary of moduli space. In this case, the intermediate gluon which connects the two sub-diagrams is on shell. In this case, the intermediate line is null, just like all of the external lines, and the contribution to the worldsheet area is small away from the cusps.

Note that there are two competing limits here: for any $\mathbf{p}^2 \neq 0$, the $e^{-\sqrt{\lambda}}$ pushes the saddle point away from the boundary of moduli space, while for $\mathbf{p}^2 \sim 0$, the pole in the propagator gives a large contribution.[16] Therefore, if we first take $\lambda \to \infty$ before taking the collinear limit $\mathbf{p}^2 = 0$, we will miss this factorization contribution.[17]

For finite values of the 't Hooft coupling, we have derived a formula for the scattering amplitude as a sum over all subdivisions of the polygonal momentum loop. There is by now some evidence that the scattering amplitude is equal to the polygonal Wilson loop in position space, both at strong coupling [2] and at weak coupling [23 – 26]. This conjecture

[15]More speculatively, this suggests a possible holographic description of off-shell amplitudes. However, we warn the reader that have not given a gauge-invariant definition of the off-shell scattering amplitude, nor have we given a reparametrization-invariant definition of the polygon momentum Wilson loop.

[16]Note that at $\mathbf{p}^2 = 0$, the factorized polygon has more cusps than the un-factorized one and is therefore more suppressed by its Sudakov form factors. However, the IR cutoff is never removed and therefore at $\mathbf{p}^2 = 0$ the pole in the intermediate propagator dominates.

[17]The mysterious factorization behavior was noticed by Chung-I Tan. We thank him for raising this question to us.

was suggested by the similarity between position and momentum loops following from T-duality in AdS [2]. However, we now see that at finite coupling, the single momentum loop cannot equal the scattering amplitude. In particular, the combination of these two formulae for the scattering amplitude (the sum over momentum polygon subdivisions and the position space polygon) would imply that the polygon in position space is equal to the sum over subdivisions of the polygon in momentum space!

We leave the further exploration of this suggestion to future work.

6. Example: Yang-Mills with adjoint Higgs

In this section we work out in an example explicit representations for the worldline integrals written more generally and abstractly in section 4. We proceed in two steps, first (in section 6.1) identifying the fields mediating the planar scattering (collectively called w in the notation of section 4) in an unambiguous way, and writing their contribution in terms of products of determinants, and then (in section 6.2) representing these determinants with worldline path integrals.

6.1 Integrating out the bifundamentals

In this section we will realize the general planar structure obtained in section 4 for a specific example. The simplest example is the scattering of gluons in pure SU(N) YM theory and we will start by describing that. However, since gluon are massless, their amplitudes are suppressed by IR-divergent Sudakov form factors. To regularize these IR divergences we will add a real adjoint Higgs field. The Higgsed theory can most simply be obtained by a KK reduction from 4+1 dimensions.[18] The Higgs will then be used to give a mass to some of the gluons and thereby to regulate these IR divergences. Note however that we will not scatter the corresponding w-bosons. These will run only on the boundary of planar diagrams and therefore will regulate the IR divergences coming from planar diagrams. The same kind of regulator was also used by Alday and Maldacena in their holographic description of the scattering amplitude [2]. In that holographic description, the massive w-boson arises from separating the finite stack of n D3 branes to which the asymptotic gluons are attached (see figure 9). This regularization has a natural description from the worldline point of view. Most studies of perturbation theory, however, use dimensional regularization. A realization of dimensional regularization in the worldline description of the gauge theory is suggested in [27].

We start from SU(N) gauge group where $n \ll N$ is the number of color indexes carried by the asymptotic gluons. We then Higgs the theory to $\mathrm{SU}(n)\times\mathrm{SU}(M)$, where $M = N - n$.

We will be interested in the large N 't Hooft limit, keeping n finite. As explained in previous 4, in that limit, loops on which the color index runs over the SU(n) part of the gauge group are down by n/N. Therefore in the planar limit, the **a** fields contribute only at tree level, whereas the **w** fields contribute only at one loop to 1PI diagrams.

[18] In this non-supersymmetric theory, in the absence of a potential for the Higgs, radiative corrections will push the Higgs vev back to zero. We ignore these below.

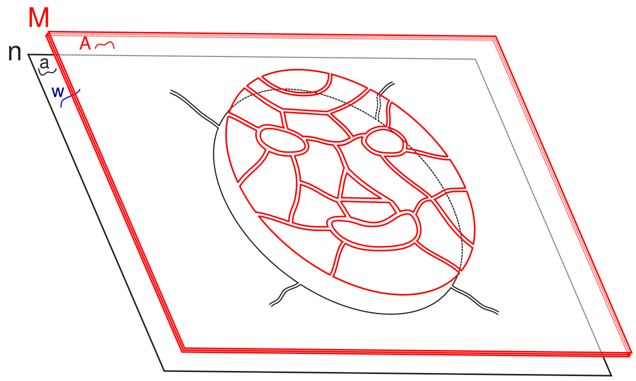

Figure 9: A generic planar diagram that contributes to the gluon scattering amplitude; the IR divergences are regularized by separating the corresponding D3-branes.

We start with the pure SU(N) YM theory of a gauge field $\widetilde{\mathbf{A}}$. To fix a gauge, we add to the Lagrangian the gauge-fixing and corresponding ghost terms

$$\mathcal{L}_{gf} + \mathcal{L}_{gh} = -\frac{1}{2}\mathrm{tr}\left(G^2\right) - \bar{c}^a \frac{\delta G^a}{\delta \theta^b} c^b \,, \tag{6.1}$$

where G is a gauge-fixing function and c are the Faddeev-Popov ghosts. To choose a gauge-fixing function that only partly fixes the gauge, we introduce the matrix

$$v = \begin{pmatrix} \mathbb{1}_{n\times n} & \\ & 0_{M\times M} \end{pmatrix} \,. \tag{6.2}$$

We then choose a Feynman-like gauge

$$G = [v, [v, \widetilde{\mathbf{D}} \cdot \mathbf{w}]]$$

where

$$\mathbf{w} = [v, [v, \tilde{\mathbf{A}}]]$$

and by taking two commutators with v we have projected on the piece that does not commute with v.

In that gauge, the gauge field splits as

$$\widetilde{\mathbf{A}} = \begin{pmatrix} \mathbf{a}_{n\times n} & \mathbf{w}_{n\times M} \\ \mathbf{w}^{\dagger}_{M\times n} & \mathbf{A}_{M\times M} \end{pmatrix} \,; \tag{6.3}$$

it will be clear from the context when fields are full $N \times N$ color matrices or their corresponding sub blocks (6.3). In this notation,

$$G = \mathbf{D} \cdot \mathbf{w} = \partial_\mu w^\mu - i\left[A_\mu + a_\mu, w^\mu\right] \,. \tag{6.4}$$

Let $\{\theta_a\}$ be the parameters representing an infinitesimal SU(n) $\times$ SU(M) gauge transformations and let $\{\widetilde{\theta}_b\}$ be the rest of the gauge parameters such that $\{\theta_a, \widetilde{\theta}_b\}$ represents a general infinitesimal SU(N) gauge transformation. The Faddeev-Popov ghost term in (6.1) represents the determinant coming from the measure in changing coordinates on the gauge orbits from $\{\theta_a, \widetilde{\theta}_b\}$ to $\{\theta_a, G_b\}$. Therefore, even though $\frac{\delta G^b}{\delta\theta^a} \neq 0$, only $\frac{\delta G^b}{\delta\widetilde{\theta}^c}$ contributes to the determinant. The resulting Faddeev-Popov ghosts are in the bifundamental of $\mathrm{SU}(n) \times \mathrm{SU}(M)$ only. To summarize, by choosing a gauge-fixing function G such that $G = [v, [v, \tilde{G}]]$, we have fixed the gauge symmetry only partly; the $\mathrm{SU}(n) \times \mathrm{SU}(M)$ subgroup is not fixed.

The Lagrangian decomposes as

$$\mathcal{L} = \mathcal{L}_{ga} + \mathcal{L}_{gf} + \mathcal{L}_{gh} \ . \tag{6.5}$$

Explicitly these are given by

$$\begin{aligned} \mathcal{L}_{ga} &= \frac{1}{4g^2}\mathrm{tr}F^2 + \frac{1}{2g^2}\mathrm{tr}\left[\mathbf{w}\cdot\left(D^2\right)\mathbf{w} + (\mathbf{D}\cdot\mathbf{w})^2 - i2w^\mu\left[F_{\mu\nu}, w^\nu\right] + [w_\mu, w_\nu][w^\mu, w^\nu]\right] \\ \mathcal{L}_{gf} &= -\frac{1}{2g^2}\mathrm{tr}\,(\mathbf{D}\cdot\mathbf{w})^2 \\ \mathcal{L}_{gh} &= -\frac{1}{g^2}\mathrm{tr}\left[\bar{c}D^2c\right] \ , \end{aligned} \tag{6.6}$$

where F and $\mathbf{D}$ are with respect to the $\mathrm{SU}(n) \times \mathrm{SU}(M)$ fields $(\mathbf{A} + \mathbf{a})$. We now add two antisymmetric auxiliary fields $d^{\mu\nu}$ and $e^{\mu\nu}$ transforming in the adjoint of $\mathrm{SU}(n)$ and $\mathrm{SU}(M)$ correspondingly. We use these to express the four w interaction as in (4.2). In the resulting action, all the dependence on the fields in the bi-fundamental of $\mathrm{SU}(n) \times \mathrm{SU}(M)$ ($\mathbf{w}$ and c) is quadratic. Integrating them out leads to the following determinants

$$\begin{aligned} \mathbf{w}: &\qquad \det\left[-D^2\eta + i2F + d + e\right]^{-\frac{1}{2}} \\ c: &\qquad \det\left[-D^2\right]^1 \ , \end{aligned} \tag{6.7}$$

where for example $F = F^a_{\mu\nu} f^{abc}$.

Next, we would like to add an adjoint Higgs field $\widetilde{\Phi}$. It decomposes into a piece that does not commute with v and a piece that does, according to:

$$\chi = [v, [v, \widetilde{\Phi}]] \ , \quad \varphi = \widetilde{\Phi} - \chi \ .$$

The resulting theory is obtained by dimensional reduction from the pure YM case described above in 4+1 dimensions, where

$$\varphi = A_4 + a_4 \quad \text{and} \quad \chi = w_4 \ .$$

The gauge fixing function that is obtained from (6.4) in 4+1 dimensions is:[19]

$$G = \mathbf{D}\cdot\mathbf{w} - i\,[\varphi, \chi] \ ,$$

We then consider the theory in a state where

$$\langle\varphi\rangle = mv \ .$$

[19]This is called the background 't Hooft-Feynman gauge in [15].

In blocks, φ decomposes as

$$\varphi = mv + \begin{pmatrix} \phi_{n\times n} & 0 \\ 0 & \Phi_{M\times M} \end{pmatrix} .$$

The reduction of the determinant (6.7) in 4+1 to 3+1 dimensions is:

$$\begin{aligned} w,\ \chi: \qquad & \det\left[\begin{pmatrix} \left(-D^2+\varphi^2\right)\eta_{\mu\nu} + 2iF_{\mu\nu} + d_{\mu\nu} + e_{\mu\nu} & 2iD_\mu\varphi + d_{4\nu} + e_{4\nu} \\ & \\ -2iD_\mu\varphi + d_{\mu 4} + e_{\mu 4} & -D^2+\varphi^2 \end{pmatrix}\right]^{-\frac{1}{2}} \\ c: \qquad & \det\left[-D^2+\varphi^2\right]^1 , \end{aligned} \tag{6.8}$$

where for example $\varphi^2 = \varphi^a\varphi^b f^{abc} f^{cde}$. Next, we will express (6.8) in the worldline formalism.

6.2 Worldline representation of determinants

Next, to evaluate the planar quantum effective action (4.7), we would like to express

$$\log\det\left(\frac{\delta^2 S}{\delta^2 w}\right)_{w=0}[\mathbf{A}+\mathbf{a}] = \log\det\left[-D^2\right] - \frac{1}{2}\log\det\left[-D^2\eta + 2iF - d - e\right] \tag{6.9}$$

in the worldline formalism. The Higgsed theory is then obtained by dimensional reduction as above. For $d = e = 0$ the corresponding worldline expression of (6.9) was derived in [15] (see [27] for a detailed review). Since d and e are anti-symmetric tensors, they can be regarded as corrections to F and the same worldline expression with $2iF \to (2iF - d - e)$ applies. If instead of introducing two anti-symmetric auxiliary tensor fields (e and d) we had introduced a single auxiliary tensor field in SU(n) that is not anti-symmetric, then in the resulting 1PI vertex we would have only auxiliary vertex insertions but no auxiliary background fields. Here, for simplicity of presentation we have chosen to introduce two anti-symmetric fields.

Below, we summarize the result of [15] for (6.9). We refer the reader to [15, 27] for further details. In appendix B we give a superspace expression for the vector determinant of [15] (at $d = e = 0$).

The first term in (6.9) is minus the contribution of a scalar field. A worldline representation of it is[20]

$$\log\det\left[-D^2\right] = -\int_0^\infty \frac{dT}{T}\mathcal{N}\int[D\mathbf{x}(\cdot)]\mathrm{tr}Pe^{i\int_0^T\left[\frac{1}{2}\dot{\mathbf{x}}^2 + \mathbf{A}\cdot\dot{\mathbf{x}}\right]} ,$$

The second term in (6.9) is the contribution of the w-boson. Following [15, 27], a worldline representation of it is

$$\begin{aligned} &-\frac{1}{2}\mathrm{Tr}\log\left[-D^2 + i2F - d - e\right] \\ &= \lim_{M\to\infty}\int_0^\infty \frac{dT}{T}\mathcal{N}\int[D\mathbf{x}(\cdot)][D\psi D\bar{\psi}]_{\mathrm{GSO}}\mathrm{tr}Pe^{i\int_0^T\frac{1}{2}\left[\dot{\mathbf{x}}^2 + i\bar{\psi}^\mu\dot{\psi}_\mu + 2\mathbf{A}\cdot\dot{\mathbf{x}} + E_{\mu\nu}\psi^\mu\bar{\psi}^\nu\right] - M^2 T} , \end{aligned} \tag{6.10}$$

[20] Note that also the gauge indices can be represented on the Hilbert space of worldline fields [28] whose string theory boundary counterparts carry the Chan Paton indices [29, 30] (see appendix B).

where

$$E_{\mu\nu} = -2iF_{\mu\nu} + d_{\mu\nu} + e_{\mu\nu} + iM^2\eta_{\mu\nu}$$

and ψ^μ is a complex fermion. The trace on the l.h.s. (6.10) is a color trace and a trace over lorentz indices. The subscript 'GSO' indicates that a sum over periodic and antiperiodic boundary conditions on the worldline fermions should be performed; this removes the states with even fermion number. The mass term $M^2\left(\eta_{\mu\nu}\bar{\psi}^\mu\psi^\nu - 1\right)$ is added to remove from the accessible spectrum the odd forms with fermion number larger than one.

We will ignore the auxiliary fields for the rest of this subsection.[21]

It may be possible to write the combination of vector (including unphysical modes) and ghost determinants as

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}][D\psi][D\bar{\psi}]\mathrm{tr}_{\mathrm{phys}} P e^{i\int_0^T \frac{1}{2}\left[\dot{\mathbf{x}}^2 + i\bar{\psi}^\mu\dot{\psi}_\mu + 2\mathbf{A}\cdot\dot{\mathbf{x}} - 2iF_{\mu\nu}\psi^\mu\bar{\psi}^\nu\right]}\,, \tag{6.11}$$

where the trace is taken over the Hilbert space of physical states. The worldline time evolution maps this space to itself. We have not verified this equality for $A \neq 0$.

We will also be interested in the worldline theory for the theory with an adjoint Higgs scalar. To add a worldline mass and a Higgs field to the physical spectrum, we simply go to five dimensions and fix the momentum in the fifth dimension to $p_4 = m$. The fifth worldline fermion ψ^4 remains and is necessary for worldline supersymmetry. The fifth component of the background gauge field becomes the Higgs field $A_4 = \Phi$. The resulting worldline representation of the YM + Higgs one loop effective action is

$$\begin{aligned}\Gamma[\mathbf{A},\Phi] \,=\, & -\frac{1}{2}\log\det\left[i\begin{pmatrix}\left(-D^2+m^2+g^2\Phi^2\right)\eta_{\mu\nu} + 2iF_{\mu\nu} & 2igD_\mu\Phi \\ -2igD_\mu\Phi & -D^2+m^2+\Phi^2\end{pmatrix}\right] \\ & + \log\det\left[i(-D^2+m^2+\Phi^2)\right] \\ = & \int_0^\infty \frac{dT}{T}\mathcal{N}\int [D\mathbf{x}(\cdot)][D\psi D\bar{\psi}]\mathrm{tr}_{\mathrm{phys}} P e^{iS[\mathbf{x},\psi,\bar{\psi};A,\Phi]}\,,\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}S = \int_0^T & \left\{\frac{1}{2}[\dot{\mathbf{x}}^2 - m^2 - \Phi^2] + i\bar{\psi}^\mu\dot{\psi}_\mu - i\psi^\mu\bar{\psi}^\nu F_{\mu\nu} + \mathbf{A}\cdot\dot{\mathbf{x}}\right. \\ & \left. + i\bar{\psi}^4\dot{\psi}_4 - i(\psi^\mu\bar{\psi}^4 - \psi^4\bar{\psi}^\mu)D_\mu\Phi\right\}.\end{aligned}$$

Background gauginos (or, turning on RR background fields). It is also interesting to consider gauge theories with fermionic fields in the adjoint, such as the gauginos of supersymmetric gauge theories. The components of these fields with the gauge quantum numbers of w are simple to include, by adding in their worldline contribution:

$$\begin{aligned}\Gamma_{\mathrm{gaugino}}[\mathbf{A}] \,&=\, \log\left[\det\left(\not{D}+m\right)\right] = \frac{1}{2}\log\left[\det\left(\not{D}+m\right)\left(-\not{D}+m\right)\right] \\ &= \frac{1}{2}\mathrm{tr}\log\left((-D^2+m^2)\mathbb{1} - \frac{i}{4}F_{\mu\nu}[\gamma^\mu,\gamma^\nu]\right)\ .\end{aligned} \tag{6.13}$$

[21]Note that the auxiliary field $d \in a$ is set to zero at the end, whereas integrating out $e \in A$ leads to a non-trivial contribution only to loops with self-crossing points.

Similarly to the scalar case, we rewrite the fermionic determinant as

$$\int_0^\infty \frac{dT}{T}\mathcal{N}\int[D\mathbf{x}][D\psi]\mathrm{tr}Pe^{i\int_0^T \frac{1}{2}\left[\dot{\mathbf{x}}^2+2i\mathbf{A}\cdot\dot{\mathbf{x}}-m^2+i\psi^\mu\dot{\psi}_\mu+iF_{\mu\nu}\psi^\mu\psi^\nu\right]}, \tag{6.14}$$

where the γ matrices are represented by real worldline fermions (ψ).

To turn on background value for the gauginos (Ψ), however, one would like to add "spin fields" into the worldline formalism. We then expect the Ramond and NS action given above to be realized in different subspaces of the Hilbert space created by the spin fields. We also expect the variation of the worldline action with respect to the gaugino to gives the worldline gaugino vertex operator. It is not clear if the worldline ghosts will still decouple. A Green-Schwarz or Berkovits-like formulation of the worldline theory (e.g. [31]) would be better for this purpose (however, it is not known how to couple these to an off-shell background).

7. Discussion

Like many of the results arising from worldline formalisms, our prescription has both perturbative and nonperturbative aspects. For example, we must sum over all tree-level diagrams in the as to retain gauge invariance, just like at any order in the perturbation expansion in a gauge theory. On the other hand, the decoupling of longitudinal polarizations follows from integration by parts, like in string theory. At strong coupling, we have tried to argue that the formula becomes more string-like, since the one-particle-irreducible contributions dominate.

In [2], AdS was shown to be self-T-dual; this suggested that the system should also have the conformal symmetry of the *dual* AdS space. Since the formulae derived in this paper give an interpretation of this T-duality as a Fourier transform in loop space, we had hoped that the formula would shed light on the mysterious 'dual conformal invariance'.

There are many approximate descriptions of scattering processes that use Wilson loops. Prominent among these are heavy quark effective field theory (e.g. [32]), and the eikonal approximation (e.g. [33, 34]). The basic idea of both is that a charged particle with enough inertia moves in a straight line, and its interactions are encoded entirely in the phase it acquires in moving through the gauge field. When applicable, a saddle point approximation to our formula should reproduce these approximations. The formula may be able to suggest a nonperturbative formulation of the inclusion of recoil corrections to these approximations; such a formulation would be useful, for example in the study of jet quenching at strong coupling [35].

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A. Reparametrization-invariant worldline theories

The worldline theories described in [15] are presented in a way that looks like the string worldsheet in conformal gauge. In addition, the vector determinant (6.9) is expressed through as a sum of two worldline path integrals, one for the w-bosons and the other for the ghosts. One would expect the sum of these two worldline theories to result from gauge fixing a single manifestly super-reparametrization invariant worldline theory.[22] For the scalar and the spinor worldline theories, a reparametrization and super-reparametrization invariant description is indeed known [36, 20]. These theories are obtained from the bosonic string and the Ramond sector of the superstring, respectively, on the annulus in the $\alpha' \to 0$ limit. In that limit all the stringy excitation decouple and the string worldsheet theory on the annulus becomes a worldline theory on the circle. Coupling the resulting worldline theory to an arbitrary background gauge field without breaking the gauge symmetries is straightforward. In these descriptions, after fixing a gauge, unphysical modes are removed by constraints. For the worldline description of the vector determinant (6.9), one would expect to obtain such super-reparametrization invariant description from the $\alpha' \to 0$ limit of the superstring in the Neveu-Schwartz sector. However, such a description is not known. Our attempts at formulating such a manifestly super-reparametrization invariant description of the NS sector were not successful; we include them as a cautionary tale for the reader. We did however find a superspace description of the 'NS sector' worldline, which may be useful and is described in appendix B.

A.1 Gauge-fixed vector worldline

A full locally-symmetric description is actually not necessary to obtain a single worldline path integral representation of the vector determinant. To see this, note that if we start with a worldline theory consist of a vector of bosons (x^μ) and a vector of complex fermions (ψ^μ) with an action

$$S = \frac{1}{2}\int d\tau \left[\dot{x}_\mu \dot{x}^\mu + i\bar{\psi}^\mu \dot{\psi}_\mu\right] ,$$

then the operators

$$Q = p_\mu \psi^\mu , \quad \bar{Q} = p_\mu \bar{\psi}^\mu , \quad \mathcal{H} = \frac{1}{2}\mathbf{p}^2$$

are conserved charges which generate a global $\mathcal{N} = 2$ symmetry. If at $\tau = 0$ we start with a physical state, then the state as well as the notion of physical does not change under worldline time evolution. That is, if at time $\tau = 0$ the state $|\varphi\rangle$ is annihilated by $\mathcal{H}$ and Q, it will continue to be upon time evolution. In addition, since $[\mathcal{H}, Q] = 0$, the physical constraint at time $\tau = 0$ is not too restrictive, so the resulting physical spectrum is not empty.

At this point this observation seems trivial, since there does exist a locally super-reparametrization invariant description of the theory (which is described in the next subsection). However, in the presence of a background gauge field we do not know how to

[22] Note that only the sum of these two worldline theories is independent of the spacetime gauge we used in section 6 to obtain the one loop determinants.

generalize that theory such that it retains its local symmetries [37], and our only consistent description of it will be as above.

The spectrum of this theory, after GSO projection,[23] contains potentials of all odd degree. To project out the unwanted three form, Strassler [15] added to the action the mass term

$$S_M = M^2 \int d\tau \left(\psi_\mu \bar{\psi}^\mu - 1\right) \ .$$

It affects only the three-form state, which is projected out in the limit $M^2 \to \infty$. In the presence of S_M, the charge Q is no longer conserved. However, since

$$[\mathcal{H}, Q] = M^2 Q \,,$$

when restricted to the kernel of $\mathcal{H}$ and Q, under time evolution, physical states remain physical and the physical spectrum is not empty.

Next, to couple the theory to a background gauge field (A), we add to the path integral the term

$$P e^{iS_A} \,, \tag{A.1}$$

where P stands for path ordering and S_A is

$$S_A = \int d\tau \left[\dot{x}_\mu A^\mu - \frac{i}{2}\psi^\mu \bar{\psi}^\nu F_{\mu\nu}\right] \ . \tag{A.2}$$

The second term in (A.2) is the supersymmetrization of the first. The resulting Hamiltonian and supercharges are

$$\mathcal{H} = \frac{1}{2}\pi_\mu \pi^\mu \,, \quad Q = \psi^\mu \pi_\mu \,, \quad \bar{Q} = \bar{\psi}^\mu \pi_\mu \,,$$

where

$$\pi_\mu = p_\mu - A_\mu = \dot{x}^\mu \ .$$

Since these close on the same algebra, when restricted to physical states at $\tau = 0$, the theory remains consistent. For non-abelian gauge theories the $[A_\mu, A_\nu]$ part of $F_{\mu\nu}$ in S_A seems to break worldline supersymmetry. However, the path ordered exponent of the worldline action is supersymetric [20]. To see this note that the SUSY variation of the terms in the worldline action linearly coupled to the external gauge field leads to a total derivative $(\partial_\tau[(\bar{\epsilon}\psi^\mu + \epsilon\bar{\psi}^\mu)A_\mu])$. When expanding the exponent of the worldline action, these gives boundary terms due to the path ordering $((\bar{\epsilon}\psi^\mu + \epsilon\bar{\psi}^\mu)[A_\mu, A_\nu]\dot{x}^\nu)$. These terms exactly cancel the SUSY variation of $\psi^\mu \bar{\psi}^\nu \, [A_\mu, A_\nu]$ coming from one lower order in the expansion.

A.2 Local super-reparametrization symmetry without background gauge field

In this subsection we describe a local $\mathcal{N} = 2$ super-reparametrization invariant worldline theory without background gauge field.

[23] The GSO projection is implemented by a sum over periodic and antiperiodic boundary conditions for the worldline fermions.

Consider the first order super-reparametrization invariant action

$$S = \int d\tau \left[p_\mu \dot{x}^\mu + \frac{i}{2}\psi_i^\mu \dot{\psi}_\mu^i - e\mathcal{H} - i\chi_i \mathcal{Q}_i \right] , \tag{A.3}$$

where

$$\mathcal{H} = \frac{1}{2}p_\mu p^\mu , \quad \mathcal{Q}_i = p_\mu \psi_i^\mu , \quad i = 1, 2 .$$

Here e, χ_1 and χ_2 are gauge fields gauging the local symmetry generated by

$$G = \alpha\mathcal{H} + i\epsilon_i \mathcal{Q}_i$$

under which the fields transform as

$$\begin{aligned} \delta x^\mu &= \alpha p^\mu + i\epsilon_i \psi_i^\mu \\ \delta p^\mu &= 0 \\ \delta\psi_i^\mu &= -\epsilon_i p^\mu \\ \delta e &= \dot{\alpha} - 2i\epsilon_i\chi_i \\ \delta\chi_i &= \dot{\epsilon}_i . \end{aligned} \tag{A.4}$$

Integrating out p^μ amounts to plugging in its equation of motion

$$p^\mu = \frac{1}{e}\left(\dot{x}^\mu - i\chi_i\psi_i^\mu\right) . \tag{A.5}$$

The resulting action is

$$S = \int d\tau \frac{1}{2e}\left[(\dot{x}_\mu - i\chi_i\psi_i^\mu)^2 + ie\psi_i^\mu \dot{\psi}_\mu^i \right] . \tag{A.6}$$

Upon quantization, the fields satisfy the canonical commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} , \quad \{\psi_i^\mu, \psi_j^\nu\} = \eta^{\mu\nu}\delta_{ij} .$$

We defined

$$\begin{aligned} \psi^\mu &= \psi_1^\mu + i\psi_2^\mu , & Q &= \mathcal{Q}_1 + i\mathcal{Q}_2 = p_\mu\psi^\mu \\ \bar{\psi}^\mu &= \psi_1^\mu - i\psi_2^\mu , & \bar{Q} &= \mathcal{Q}_1 - i\mathcal{Q}_2 = p_\mu\bar{\psi}^\mu . \end{aligned}$$

These satisfy the commutation relations

$$\{Q, \bar{Q}\} = 2\mathcal{H} , \quad [\mathcal{H}, Q] = [\mathcal{H}, \bar{Q}] = 0 .$$

Next, we quantize the theory in the Gupta-Bleuler style by imposing that physical states are annihilated by $\bar{Q}$ and $\mathcal{H}$. At each momentum p, the Hilbert space is spanned by the states

$$\begin{aligned} &|0, p\rangle , && \varepsilon_\mu\psi^\mu|0, p\rangle , && f_{\mu\nu}\psi^\mu\psi^\nu|0, p\rangle \\ &g_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho|0, p\rangle , && \psi^0\psi^1\psi^2\psi^3|0, p\rangle . \end{aligned}$$

By GSO projection (i.e. gauging the worldline fermion number), we project to the states

$$\varepsilon_\mu \psi^\mu |0,p\rangle \quad \text{and} \quad g_{\mu\nu\rho}\psi^\mu\psi^\nu\psi^\rho|0,p\rangle$$

only. The Hamiltonian constraint is

$$p^2 = 0$$

and the $\bar{Q}$ constraint reads

$$p^\mu \varepsilon_\mu = 0\,, \quad p^\mu g_{\mu\nu\rho} = 0\ .$$

The remaining states are a transverse one form and transverse three form. After the physical projection, there are no negative norm states in the physical spectrum.

Now consider the addition of S_A, the coupling to the background gauge field. With the supersymmetry transformations (A.4), this term is not invariant. To make this term invariant will require some nonlinear realization of the worldline local supersymmetry, which is beyond the scope of this paper.

B. String-inspired and superspace worldline theories

The worldline theories in question can in principle be obtained in a very direct way from string theory: they describe the zero-slope limit of an open type IIB superstring stretched between two parallel D3-branes. Using the RNS formalism for this string worldsheet in conformal gauge, we obtain the spinors from the R sector and the vectors and scalars from the NS sector. Finally, we give a superspace representation of the vector (NS) worldline theory coupled to background gauge field. We ignore the auxiliary fields introduced in sections 4 and 6 for this discussion.

B.1 Dimensional reduction of the open string

In the NS sector of an open superstring, on the doubled strip ($\sigma \in [0, 2\pi)$), the fermions have boundary conditions

$$\psi^\mu(\sigma^1 + 2\pi, \sigma^2) = -\psi^\mu(\sigma^1, \sigma^2)\ .$$

The general field with such boundary condition can be expanded as

$$\psi^\mu(\sigma^1, \sigma^2) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} \psi_r(\sigma^2) e^{ir\sigma^1}\ ;$$

note that we have not solved the equations of motion, only the boundary condition. Plugging into the action, this gives

$$\begin{aligned} S[\psi] &= \int d\sigma^1 d\sigma^2\ \ \frac{1}{2}\eta_{\mu\nu} i\psi^\mu \left(\partial_2 - \partial_1\right)\psi^\nu \\ &= \int d\sigma^2 \sum_{r \in \mathbb{Z}+\frac{1}{2}} \frac{1}{2}\psi_r \left(i\partial_2 + r\right)\psi_{-r} = \int d\sigma^2 \sum_{r=\frac{1}{2},\frac{3}{2}\ldots>0} \psi_r \left(i\partial_2 + r\right)\psi_{-r}\ . \end{aligned}$$

We will truncate this set of modes to only the lowest-lying conjugate pair $r = \pm\frac{1}{2}$; only these modes create massless states. The masses of the other states are larger by an amount proportional to the string scale which we will take to ∞.

The worldsheet gravitino has a spin structure which is correlated with that of the RNS fermions:

$$\chi(\sigma^1 + 2\pi, \sigma^2) = -\chi(\sigma^1, \sigma^2) .$$

Reducing the coupling to the supercurrent gives

$$\int d\sigma^1 d\sigma^2 \ \chi T_F = \int d\sigma^2 \left(\chi_{\frac{1}{2}} G_{-\frac{1}{2}} + \chi_{-\frac{1}{2}} G_{\frac{1}{2}} + \dots \right) .$$

The worldline action (in units where $\frac{2\pi}{4\pi\alpha'} = \frac{1}{2}$) is

$$S = \int dt \left(\frac{1}{2}\dot{\mathbf{x}}^2 + \bar{\psi}_\mu \left(i\partial_t + \mu \right) \psi^\mu \right) ,$$

where we have changed the name of the worldline time $\sigma^2 = t$ and $\psi \equiv \psi_{-\frac{1}{2}}, \bar{\psi} \equiv \psi_{\frac{1}{2}}$, and $\mu = \frac{1}{2}$. Notice that supersymmetry is 'broken,' in the sense that bosonic and fermionic states related by the action of G do not have the same energies — of course, this is a consequence of the NS boundary condition. The constraint algebra, however, still closes and is generated by

$$\begin{aligned} G_{-\frac{1}{2}} \equiv Q = \psi^\mu p_\mu, \quad & G_{\frac{1}{2}} \equiv \bar{Q} = \bar{\psi}^\mu p_\mu \\ H = \frac{1}{2}p^2 + \mu J, & \end{aligned}$$

where $J \equiv \psi\bar{\psi}$ is the fermion number operator $[J, \psi] = \psi, [J, \bar{\psi}] = -\bar{\psi}$. The worldline algebra (before coupling to a background gauge field) is

$$\frac{1}{2}\{Q, \bar{Q}\} = H - \mu J, \quad Q^2 = 0 = \bar{Q}^2, \quad [H, Q] = \mu Q, \quad [H, \bar{Q}] = -\mu\bar{Q} . \tag{B.1}$$

In the same way that in old covariant quantization of the superstring, the modes of the supervirasoro algebra are treated as Gupta-Bleuler constraints,

$$G_{r>0}|\text{phys}\rangle = 0$$

(since this is enough to guarantee that all physical-state matrix elements of the constraint algebra generators vanish), we should only impose that

$$\bar{Q} \equiv G_{\frac{1}{2}}$$

annihilate physical states.

In the superstring, the vacuum energy must take a definite value to be consistent with conformal invariance. This value is such that states with a single $\psi_{-\frac{1}{2}}$ excitation are massless. We impose this value of the vacuum energy in our worldline theory.

The worldline theory also inherits the GSO projection onto states with one parity of the fermion number. That is the parity such that the massless states survive. Note that the worldsheet ghosts carry GSO-charge; if we were careful about them in the worldline theory (which would clearly be best done simply by dimensionally reducing the NS sector ghosts of the superstring $b_0, c_0, \gamma_{\pm\frac{1}{2}}, \beta_{\pm\frac{1}{2}}$), we could see that the NS ground state has odd fermion number.

If we perform the same dimensional reduction in the Ramond sector, we obtain the usual spinning particle of [36]. The GSO projection, consistent in even numbers of dimensions, makes the resulting spinor chiral.

Adding a spacetime mass. So far we have discussed the dynamics of the coordinates *along* a Dp brane (and therefore $\mu = 0, \ldots, p$). The directions perpendicular to the brane are obtained from the above simply by setting their momentum $p_i = 0$, $A_i = 0$ where $i = p+1, \ldots, d$. Of course, the dimensions we are most interested in are $p = 3$ and $d = 9$. However, the worldline theory seems to be consistent for any number of dimensions.

Next, we would like to Higgs the theory. From the point of view of the string theory, that can be accomplished by separating the D-branes. We can then redo the dimensional reduction in a sector with nonzero winding:

$$x^4(\sigma, t) = \Delta y \sigma \tag{B.2}$$

but again eliminate all oscillator modes. In the R-sector, this procedure reproduces the massive spinning particle [36]. The mass m appearing there is

$$m = \frac{\Delta y}{\alpha'} \tag{B.3}$$

and is kept fixed in the zero slope $\alpha' \to 0$ limit. In the NS sector, the resulting supercharges (ignoring the terms involving CP fields which are unaffected) are

$$Q = \psi_\mu \dot{x}^\mu + m\psi^4\,, \quad \bar{Q} = \bar{\psi}_\mu \dot{x}^\mu + m\bar{\psi}^4\,, \tag{B.4}$$

where $\psi^4 \equiv \psi^4_{-\frac{1}{2}}, \bar{\psi}^4 \equiv \psi^4_{\frac{1}{2}}$.

Finally, we must discuss the Faddeev-Popov ghosts for the vector case. They again decouple from the worldline action of the matter fields (x, ψ, η). Their partition function is of the form

$$Z_{gh} = \int [Db_0 Dc_0 D\beta_{\frac{1}{2}} D\beta_{-\frac{1}{2}} D\gamma_{\frac{1}{2}} D\gamma_{-\frac{1}{2}}] \;\; e^{iS_{\rm gh}}. \tag{B.5}$$

B.2 Superspace representation of the worldline theory

Having a superspace representation of a supersymmetric theory is a very useful technical tool. The spinor worldline theory has an $\mathcal{N} = 1$ superspace description [36]. Similarly, one would like to have an $\mathcal{N} = 2$ superspace description of the vector theory described in [15] and summarized in section 6. However, the presence of a mass for the worldline fermions manifestly break the global $\mathcal{N} = 2$ supersymmetry. Here we give a deformed $\mathcal{N} = 2$ superspace representation of it. The ability to write down such a superspace description is related to the fact that at the level of the action (but not the path integral measure) a mass (μ) for the worldline fermions can be absorbed in a field redefinition $\psi \to e^{i\mu t}\psi$.

Define

$$D_0 \equiv \partial_\theta + \bar\theta i\partial_t, \qquad \bar D_0 \equiv \partial_{\bar\theta} + \theta i\partial_t \ ;$$

these are the $\mathcal{N}=2$ superspace derivatives with $\mu = 0$, i.e. the ordinary ones. $\mathcal{N}=2$ superspace derivatives which generate the truncated NS algebra are

$$D = e^{i\mu t}\left(D_0 + \mu\theta\bar\theta\partial_\theta\right) \qquad \bar D = e^{-i\mu t}\left(\bar D_0 + \mu\theta\bar\theta\partial_{\bar\theta}\right).$$

They satisfy

$$\begin{aligned} D^2 &= 0, \qquad \bar D^2 = 0 \\ \{D, \bar D\} &= 2i\partial_t - 2\mu\left(\theta\partial_\theta - \bar\theta\partial_{\bar\theta}\right) = -2\left(H + \mu J\right) \ . \end{aligned}$$

The fact that $D^2 = 0$ is made manifest by the observation that D can be rewritten as

$$D = e^{i\mu\bar y} D_0, \qquad \bar D = e^{-i\mu y}\bar D_0 \,,$$

where y is the chiral time coordinate

$$y \equiv t + i\theta\bar\theta$$

which satisfies $\bar D y = 0$ (and $\bar D_0 y = 0$).

Each of the worldline supercoordinates X^μ comprises a real superfield:

$$X = x + \theta\psi + \bar\theta\bar\psi + \theta\bar\theta F \ .$$

Also, we will need to be more explicit about the gauge representation of the worldline. To do this it is useful to introduce worldline degrees of freedom η whose hilbert space generates the Chan-Paton space [28]; from the point of view of the zero-slope open string, these are boundary degrees of freedom [29, 30]. We will follow Friedan and Windey [38].

The modes generating the CP $\underline{\eta}^{a=1\ldots N}$ will be chiral superfields whose lowest components are fermions (fermi multiplet):

$$\bar D\underline{\eta} = 0 \quad \Longrightarrow \quad \underline{\eta} = \eta(y) + \theta b(y)$$

where the components are functions of the chiral time coordinate y.

A reasonable lagrangian which contains kinetic terms for X and η is

$$\frac{1}{2}\int d^2\theta(DX\bar DX + \bar\eta\eta) = \frac{1}{2}\dot x^2 + i\bar\psi\dot\psi - \mu\bar\psi\psi + \frac{F^2}{2} + i\bar\eta\dot\eta - \frac{1}{2}\bar b b \ ; \tag{B.6}$$

here we define $\int d^2\theta \equiv \frac{1}{2}[\bar D, D]$. The similarity between the fields generating the CP space and the worldline fermions has often been remarked upon; it is a worldline remnant of the fact that their 2d avatars both generate an SU(N) current algebra. The supercharges are

$$Q = \psi\dot x + i\bar\eta b\,, \qquad \bar Q = \bar\psi\dot x - i\eta\bar b \ . \tag{B.7}$$

The states of the η's generate $2^N = \oplus_{p=0}^N \wedge^p \mathbf{N}$ states. We will be interested mainly in the case where the worldline transforms in the fundamental $\mathbf{N}$ of the U(N) group. To accomplish the projection onto this irrep, we introduce a worldline gauge field a_0, which is a supersymmetry singlet. We add a 1d Chern-Simons term $\Delta S = q\int dt\, a_0$ and couple a_0 to the η number current. The terms involving a_0 are then

$$\int dt\, a_0\, (\eta\bar{\eta} - 1) \ . \tag{B.8}$$

This action is supersymmetric on-shell (i.e. after using the equations of motion for b);[24] this is because the η-number current commutes with the supercharges (B.7) on-shell. If for some reason one want to generate a symmetric-tensor representation of $\mathrm{U}(N)$, one should introduce bosonic chiral multiplets on the worldline.

Coupling to a background gauge field. The superspace expression on the l.h.s. of (B.6) above includes the coupling to the gauge field if one promotes the action of D on η to the covariant one D_A as in Windey et al. We modify the superspace derivatives when they act on charged fields to

$$D \longrightarrow D_A = D + iDX^\mu A_\mu(X)\,, \qquad \bar{D} \longrightarrow \bar{D}_A = \bar{D} + i\bar{D}X^\mu A_\mu(X)\ .$$

The algebra is

$$\begin{aligned}\{D_A, \bar{D}_A\} &= \{D + iDXA, \bar{D} + i\bar{D}XA\} \\ &= i\partial_t + \{iDXA, \bar{D}\} + \{D, i\bar{D}XA\} - \{DXA, \bar{D}XA\} \\ &= i\partial_t + i\left(\{\bar{D}, D\}X\right) A + iDX\bar{D}A + i\bar{D}XDA - DX^\mu\bar{D}X^\nu[A_\mu, A_\nu] \\ &= i\partial_t - \partial_t X^\mu A_\mu + iDX^\mu\bar{D}X^\nu F_{\mu\nu}\,.\end{aligned}$$

Similarly

$$D_A^2 = iDX^\mu DX^\nu F_{\mu\nu}\,, \qquad \bar{D}_A^2 = i\bar{D}X^\mu\bar{D}X^\nu F_{\mu\nu}\ .$$

We now demand that η is covariantly chiral: $\bar{D}_A\eta = 0$. The kinetic terms for $\eta^{a=1\ldots N}$ become

$$\begin{aligned}\frac{1}{2}\int d^2\theta\bar{\eta}_a\eta^a &= -\frac{1}{2}\bar{D}_A\bar{\eta}_a D_A\eta + \frac{1}{4}\left(\{D_A, \bar{D}_A\}\bar{\eta}_a\right)\eta^a - \frac{1}{4}\bar{\eta}_a\left(\{\bar{D}_A, D_A\}\eta^a\right) \\ &= \bar{\eta}\left(i\partial_t - \dot{x}^\mu A_\mu + i\psi^\mu\bar{\psi}^\nu F_{\mu\nu}\right)\eta - \frac{1}{2}\bar{b}b\ .\end{aligned} \tag{B.9}$$

Since the NS mass ($\mu = \frac{1}{2}$) affects only the higher components of superfields, it has no effect on (B.9). Note that η and b appearing in the previous equation are defined as

$$\eta \equiv \underline{\eta}|_{\theta=\bar{\theta}=0}\,, \qquad b \equiv e^{-i\mu t} D_A\underline{\eta}|_{\theta=\bar{\theta}=0}.$$

[24] An off-shell supersymmetric version of the first term in (B.8) is

$$\int d^2\theta \int dt\, \bar{\eta}(t)\, a_0(t) \int^t dt'\, \eta(t')\ .$$

In checking this, it is convenient to choose one-dimensional Lorentz gauge: $\partial_t a_0 = 0$. Note, however, that this expression is not gauge invariant. The one-dimensional Chern-Simons term $q\int dta_0$ is supersymmetric, and gauge invariant as long as $q \in \mathbb{Z}$.

Integrating out the auxiliary fields b by their algebraic equations of motion gives $b = 0 = \bar{b}$. The covariant supercharges expressed as differential operator in superspace is

$$\begin{aligned} Q_A &= e^{-i\mu y}\left(\partial_\theta - i\bar{\theta}\partial_t\right) + i\left[Q_0, X^\mu\right] A_\mu(X) \\ \bar{Q}_A &= e^{i\mu\bar{y}}\left(\partial_{\bar{\theta}} - i\theta\partial_t\right) + i\left[\bar{Q}_0, X^\mu\right] A_\mu(X)\,, \end{aligned} \tag{B.10}$$

where A is a color matrix acting only on η and μ act only on X. The Noether currents associated to the corresponding supersymmetry transformations are[25]

$$\begin{aligned} Q &= e^{-i\mu t}\left[\psi^\mu \dot{x}_\mu + i\bar{\eta}b\right] = e^{-i\mu t}\left[\psi^\mu p_\mu + i\bar{\eta}\left(b + i\psi^\mu A_\mu \eta\right)\right] \\ \bar{Q} &= e^{i\mu t}\left[\bar{\psi}^\mu \dot{x}_\mu + i\eta\bar{b}\right] = e^{i\mu t}\left[\bar{\psi}^\mu p_\mu + i\left(\bar{b} + i\bar{\eta}\bar{\psi}^\mu A_\mu\right)\eta\right] \ . \end{aligned} \tag{B.11}$$

Note that these are the non-covariant supercharges. Their commutation relation with H are also obtained from the Noether theorem as

$$\delta_{\epsilon Q} S = i\int dt\epsilon\,[H,Q] = i\int dt\,\epsilon\mu Q\,, \tag{B.12}$$

which is true for any epsilon and hence reproduces (B.1). Finally, we have

$$\{Q,\bar{Q}\} = (\mathbf{p} + \bar{\eta}\mathbf{A}\eta)^2 + i\psi^\mu\bar{\psi}^\nu F_{\mu\nu} = 2\,(H - \mu J)\ ,$$

where H is the operator that generates the non-gauge-covariant time derivative ($[H,\mathcal{O}] = -i\partial_t\mathcal{O}$) and we have set $b = 0$. The supercharges commute with the η-number current on-shell, so the term (B.8) is supersymmetric.[26]

Note that it is the non-covariant hamiltonian and supercharges that generate a symmetry of the worldline action coupled to a fixed background gauge field. Therefore, it is these charges that are gauged on the worldline. Once we include the path integral over the background gauge field, also the covariant hamiltonian generates a symmetry (which is gauged).

C. Wilson sums and the loop operator

In [5], A. Polyakov suggested a relation between the gauge theory loop operator for momentum Wilson loops and the dual string theory Virasoro generator. The action of the loop operator on Migdal's momentum Wilson loop for the polygon (3.4) is [9][27]

$$\hat{L}\widetilde{W}[\mathbf{p}(\cdot)]_{\mathrm{Migdal}} = \sum_i \mathbf{k}_i^2\ \widetilde{W}[\mathbf{p}(\cdot)]_{\mathrm{Migdal}}\ . \tag{C.1}$$

Polyakov interpreted the momentum Wilson loop evaluated on the polygon as describing an open string stretched between D-instantons (which are the T-dual to the D3-branes) located at the positions $\{\mathbf{p}_j = \sum_{i\leq j}\mathbf{k}_i\}$ and the loop operator as the integrated Virasoro generator L_0 in the $\alpha' \to 0$ limit.[28]

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[25] For convenience, we have rescaled the supercharges by a factor of i.

[26] Since (B.8) is neutral, the covariant and the non-covariant supercharges act on it in the same way.

[27] We remind the reader that $\widetilde{W}[\mathbf{p}(s)]_{\mathrm{Migdal}} \neq \widetilde{W}[\mathbf{p}(s)]$.

[28] The loop equation then equates (C.1) to a sum over divisions of $W[\mathbf{p}(s)]_{\mathrm{Migdal}}$ into two sub-loops.

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In the previous sections we have expressed scattering amplitudes in terms of Wilson sums. Next, to obtain an analog of (C.1), we will act with the loop operator on the 1PI scalar vertex written in terms of a Wilson sum.

The loop operator is a differential operator acting on functionals of closed loops. One representation of it is

$$\hat{L} \equiv \lim_{\epsilon \to 0} \oint d\tau \int_{-\epsilon}^{\epsilon} ds \frac{\delta^2}{\delta \mathbf{x}(\tau) \cdot \delta \mathbf{x}(\tau + s)} \ . \tag{C.2}$$

Even though it is expressed in terms of some parametrization of the loop $\mathbf{x}(s)$, $\hat{L}$ is reparametrization invariant and therefore has a well defined action on loop space functionals.

As defined in (C.2), $\hat{L}$ is the natural loop operator that acts on Wilson loops with standard coupling to the background gauge field

$$\oint \mathbf{A} \cdot d\mathbf{x} \ .$$

These are the building blocks of the scalar contributions to the 1PI vertex. Without using the trick (4.11), the w-boson contribution also has the coupling

$$\int ds \psi^{\mu} \bar{\psi}^{\nu} F_{\mu\nu} \ .$$

The corresponding loop operator has additional terms involving the worldline fermions [10, 11]. Here, for simplicity, we consider the action of the loop operator on the scalar 1PI vertex.

The scalar 1PI vertex $A_n^{\mathbf{P}}$ (2.2), is not a functional of position space loops but a function of the external momenta and polarization. However, it is given in terms of a sum over closed loops. Each closed loop in that sum is weighted by a product of functionals. One is the Wilson loop expectation value $\langle W[\mathbf{x}(\cdot)]\rangle$, another is the functional

$$F[\mathbf{x}(\cdot)] \equiv \prod_{i=1}^{n} \int_{s_{i-1}}^{T} ds_i \ \varepsilon_i \cdot \dot{\mathbf{x}}(s_i) \ e^{i\mathbf{k}_i \cdot \mathbf{x}(s_i)} \quad ,$$

and the rest can be attributed to the measure.

Next we act with the loop operator on the Wilson loop expectation value and perform an integration by parts in the integral over closed loops to obtain a relation analogous to (C.1):

$$\begin{aligned}
&\int \frac{dT}{T} \int [D\mathbf{x}(\cdot)] \, e^{-\int_0^T \frac{1}{2}\dot{\mathbf{x}}^2 ds} F[\mathbf{x}(\cdot)] \ \hat{L} \langle W\left[\mathbf{x}(\cdot)\right]\rangle \\
&= \int \frac{dT}{T} \int [D\mathbf{x}(\cdot)] \, e^{-\int_0^T \frac{1}{2}\dot{\mathbf{x}}^2 ds} F[\mathbf{x}(\cdot)] \ \langle W\left[\mathbf{x}(\cdot)\right]\rangle \sum_l \left[\Delta \dot{\mathbf{x}}(\tau_l) + i\Delta \mathbf{p}(\tau_l)\right]^2 \\
&\quad + \int \frac{dT}{T} \int [D\mathbf{x}(\cdot)] \, e^{-\int_0^T \frac{1}{2}\dot{\mathbf{x}}^2 ds} \langle W\left[\mathbf{x}(\cdot)\right]\rangle \left(\prod_{i=1}^{n} \int_{s_{i-1}}^{T} ds_i \ e^{i\mathbf{k}_i \cdot \mathbf{x}(s_i)} \right) \sum_{j=1}^{n} \left(\prod_{k \neq j} \varepsilon_k \cdot \dot{\mathbf{x}}(s_k) \right) \\
&\quad \times \varepsilon_j \cdot \left[\Delta \dot{\mathbf{x}}(s_j) + \Delta \mathbf{p}(s_j)\right] \left[\delta(s_j - s_{j+1}) - \delta(s_j - s_{j-1}) - i\mathbf{k}_j \cdot \dot{\mathbf{x}}(s_j)\right] \ ,
\end{aligned} \tag{C.3}$$

where for any wordline function $\mathbf{y}$,

$$\Delta\mathbf{y}(\tau) \equiv \mathbf{y}(\tau+0) - \mathbf{y}(\tau-0)$$

is the discontinuity of $\mathbf{y}(s)$ at $s = \tau$ and $\{\tau_l\}_l$ are the points were $\dot{\mathbf{x}}$ or $\mathbf{p}$ have a discontinuity. $\mathbf{p}(s)$ is given in (3.4). With these definitions, $\Delta\mathbf{p}$ is only nonzero when $\tau = s_i$ is the location of a vertex insertion, in which case $\Delta\mathbf{p} = \mathbf{k}_i$. In the case that vertices collide, $s_i = s_{i+1}$, the jumps in momenta add; altogether

$$\Delta\mathbf{p}(\tau) = \begin{cases} \mathbf{k}_i & \tau = s_i \in (s_{i-1}, s_{i+1}) \\ \mathbf{k}_i + \mathbf{k}_{i+1} & \tau = s_i = s_{i+1} \\ \vdots & \vdots \\ 0 & \text{otherwise} \end{cases}$$

The differences between (C.3) and (C.1) come from the difference in the measures (which gives the $\Delta\dot{\mathbf{x}}$ terms) and the polarization dependence of the gluon vertex $\varepsilon\cdot\mathbf{x}$ (which gives the last line in (C.3)). The loop equation can be further used to express $\hat{L}\langle W[\mathbf{x}(\cdot)]\rangle$ as a sum of the self-crossing points of $\mathbf{x}(\cdot)$ times the product of the two sub-loops (see for example [9]).

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